Exercise 1, Functional Equations. Find all $f : \mathbb{R} \to \mathbb{R}$ that satisfy $f(x)f(y) - f(x+y) = x + y$ for all $x, y \in \mathbb{R}$.

Solution, by Ralph Furmaniak. Let $x = y = 0$. Then

$$\begin{aligned} f^2(0) - f(0) &= 0 \\ f(0)(f(0) - 1) &= 0, \end{aligned}$$

so $f(0) = 0$ or 1. If $f(0) = 0$, then when we let $y = 0$, we get

$$\begin{aligned} f(x)f(0) - f(x) &= x \\ f(x) &= -x. \end{aligned}$$

When we plug this function into the given functional equation, we get

$$\begin{aligned} (-x)(-y) + x + y &= x + y \\ xy &= 0. \end{aligned}$$

Since this is not always true, we conclude that $f(0) \neq 0$, so $f(0) = 1$. But then, again letting $y = 0$, we get

$$f(x) - f(x) = x,$$

which again is not always true. Thus, there are no equations that satisfy the given functional equation. □