The Second Annual UNLV Mathematics Competition

Saturday, November 11th, 9:00 am to 12:30 pm. To receive full marks, solutions must be complete and well justified. Decisions of the judges are final. Solutions will be posted at http://www.nevada.edu/∼baragar

1. Equilateral triangles ΔABP and ΔACQ are constructed on the exterior of a triangle ΔABC. Prove that the lengths $|CP|$ and $|BQ|$ are equal.

Solution 1. Rotate ΔABQ about A through an angle of $60°$ clockwise (see Figure 1). Then Q goes to C and B goes to P. Hence, $|CP| = |BQ|$. □

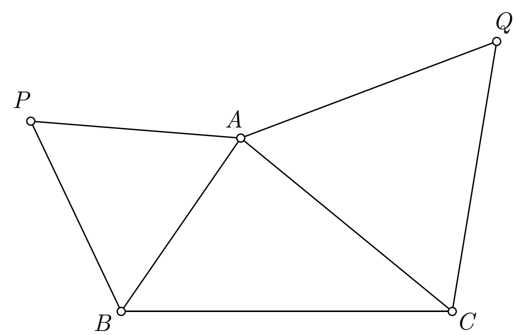

Figure 1

Solution 2. Consider ΔABQ and ΔAPC. Note that $\angle BAQ = A + 60° = \angle PAC$. Also, $|AB| = |AP|$ and $|AQ| = |AC|$. Thus, by SAS, ΔABQ and ΔAPC are congruent. In particular, $|BQ| = |PC|$. □

2. For each parabola $y = x^2 + px + q$ which meets the coordinate axes in three points, consider the circle through these three points. Show that all these circles pass through a common point.

Solution. Let r_1 and r_2 be the roots of

$$f(x) = x^2 + px + q.$$

Then, the parabola intersects the axes at $(r_1, 0)$, $(r_2, 0)$, and $(0, q)$ (see Figure 2). Let (a, b) be the center of the circle which goes through these three points.

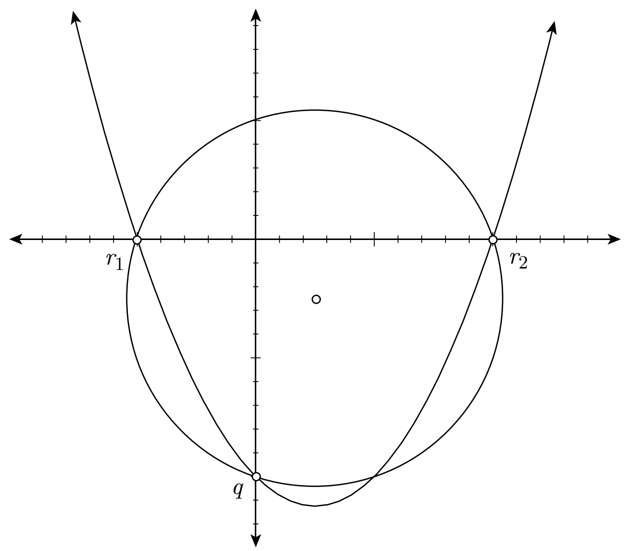

Figure 2

Then (a,b) is on the perpendicular bisector of the chord from $(r_1,0)$ to $(r_2,0)$. That is,

$$a = \frac{r_1+r_2}{2} = \frac{-p}{2}.$$

Since $(r_1,0)$ and $(0,q)$ are equidistant from $(a,b) = \left(\frac{-p}{2}, b\right)$, we get

$$\left(\frac{-p}{2} - r_1\right)^2 + b^2 = \left(\frac{-p}{2}\right)^2 + (b-q)^2$$
$$r_1^2 + r_1 p + \frac{p^2}{4} + b^2 = \frac{p^2}{4} + b^2 - 2bq + q^2.$$

Since r_1 is a root of $f(x)$, we know $r_1^2 + r_1 p = q$, so

$$-q = -2bq + q^2.$$

Note that if $q = 0$, then there are at most two points of intersection between the parabola and the axes, so we know $q \neq 0$. Thus,

$$-1 = -2b + q$$
$$b = \frac{q+1}{2}.$$

Hence, the circle has the equation

$$\left(x+\frac{p}{2}\right)^2+\left(y-\frac{q+1}{2}\right)^2=\left(\frac{p}{2}\right)^2+\left(q-\frac{q+1}{2}\right)^2$$
$$=\left(\frac{p}{2}\right)^2+\left(\frac{q-1}{2}\right)^2.$$

By inspection, we note that this always has the solution $(0,1)$. Thus, all such circles go through the point $(0,1)$. □

3. Two sets A and B are called *almost disjoint* if they have at most one element in common. Find the size of the largest family of almost disjoint subsets of $\{1,2,3,...,n\}$.

Solution. Let F be a maximal family of almost disjoint subsets of $\{1,2,...,n\}$. Suppose there exists an element S_0 of F such that

$$S_0=\{a_1,...,a_k\}$$

with $k\geq 3$. We will show that, if F contains such an element, then F is not maximal. To see this, let us construct

$$F'=\{S\in F : S\neq S_0\}\cup\{\{a_1,a_k\},\{a_1,...,a_{k-1}\}\}.$$

Then F' contains one more element than F. Let us now show that F' is a family of almost disjoint subsets of $\{1,...,n\}$. Let S and T be two elements of F'. Then there are a number of possibilities: Either S and T are also in F, so they are almost disjoint; or S is in F and $T=\{a_1,a_k\}$, in which case

$$S\cap T\subset S\cap S_0,$$

and since the latter intersection contains at most one element, S and T are almost disjoint; or S is in F and $T=\{a_1,...,a_{k-1}\}$, in which case S and T are almost disjoint using the same argument; or $S=\{a_1,a_k\}$ and $T=\{a_1,...,a_{k-1}\}$, in which case $S\cap T=\{a_1\}$, so they are almost disjoint. Hence, F' is a family of almost disjoint subsets of $\{1,...,n\}$, and hence, F is not maximal (since $|F'|=|F|+1$). Thus, if F is maximal, then every element of F contains at most two elements of $\{1,...,n\}$. Note that the family F'' which contains all subsets of $\{1,...,n\}$ which have only 0, 1, or 2 elements is in fact a family of almost disjoint subsets, since if the intersection of any two such subsets contains two elements, then the two subsets are equal. Thus, the maximal family is F'', which contains

$$|F''| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} = 1 + n + \frac{n(n-1)}{2} = \frac{n^2+n+2}{2}$$

elements. □

4. Find all pairs (a, b) of positive integers such that the equation

$$x^3 - 17x^2 + ax - b^2 = 0$$

has three integer roots (not necessarily distinct).

Solution. Let

$$f(x) = x^3 - 17x^2 + ax - b^2.$$

Suppose $r \geq 0$. Then

$$f(-r) = -r^3 - 17r^2 - ar - b^2 < 0.$$

Hence, all roots of $f(x)$ are positive. Let these roots be r, s and t, and let us order these roots so that

$$0 < r \leq s \leq t.$$

Then

$$f(x) = (x-r)(x-s)(x-t) = x^3 - (r+s+t)x^2 + (rs+rt+st)x - rst.$$

Hence,

$$\begin{aligned} r + s + t &= 17 \\ rst &= b^2. \end{aligned}$$

There are only a finite number of possibilities for (r, s, t) in the integers, so let us check them.

If $r = 1$, then $s + t = 16$, so $(s, t) = (1, 15)$, $(2, 14)$, $(3, 13)$, $(4, 12)$, $(5, 11)$, $(6, 10)$, $(7, 9)$, or $(8, 8)$. Of these, only $(8, 8)$ makes st a perfect square.

If $r = 2$, then $s + t = 15$ so $(s, t) = (2, 13)$, $(3, 12)$, $(4, 11)$, $(5, 10)$, $(6, 9)$, or $(7, 8)$. Of these, only $(5, 10)$ makes $2st$ a perfect square.

If $r = 3$, then $s + t = 14$, so $(s, t) = (3, 11)$, $(4, 10)$, $(5, 9)$, $(6, 8)$, or $(7, 7)$. Of these, only $(6, 8)$ makes $3st$ a perfect square.

If $r = 4$, then $s + t = 13$, so $(s, t) = (4, 9)$, $(5, 8)$, or $(6, 7)$. Of these, only $(4, 9)$ makes $4st$ a perfect square.

If $r = 5$, then $s + t = 12$, so $(s, t) = (5, 7)$ or $(6, 6)$. Neither of these makes $5st$ a perfect square.

Finally, if $r \geq 6$, then $s + t \leq 11$, so $s < r$. Hence, there are no more to check.

Thus, the only solutions are $(1, 8, 8)$, $(2, 5, 10)$, $(3, 6, 8)$, and $(4, 4, 9)$. These give $(a, b) = (80, 8)$, $(80, 10)$, $(90, 12)$, and $(88, 12)$, respectively. □

5. Let b_n be the number of ways that n can be written in the form

$$n = a_0 + a_1 2 + a_2 2^2 + ... + a_m 2^m = \sum_{k=0}^{m} a_k 2^k$$

where m is arbitrary, $a_k \in \{0, 1, 2\}$, and $a_m \neq 0$. What is b_{2000}? (Hint: Think of recurrence relations. I.e., think of b_n in terms of b_k's for smaller k.)

Proof. Consider a representation for an odd number $2n + 1$. It is clear that $a_0 = 1$, so

$$\begin{aligned} 2n + 1 &= 1 + a_1 2 + ... + a_m 2^m \\ &= 1 + 2(a_1 + a_2 2 + ... + a_m 2^{m-1}). \end{aligned}$$

Hence, every representation of $2n + 1$ can be derived from a representation of n. That is,

$$b_{2n+1} = b_n.$$

Now, let us consider a representation of an even number $2n$. Then $a_0 = 0$ or 2. If $a_0 = 0$, then

$$\begin{aligned} 2n &= a_1 2 + ... + a_m 2^m \\ &= 2(a_1 + a_2 2 + ... + a_m 2^{m-1}). \end{aligned}$$

Thus, every such representation can be derived from a representation of n. If $a_0 = 2$, then

$$2n = 1 + (1 + a_1 2 + ... + a_m 2^m),$$

so every such representation can be derived from a representation of $2n-1$. Thus,

$$b_{2n} = b_n + b_{2n-1} = b_n + b_{n-1}.$$

To get us started, we note that $b_0 = b_1 = 1$. Thus,

$$\begin{aligned}
b_{2000} &= b_{1000} + b_{999} \\
&= b_{500} + b_{499} + b_{499} = b_{500} + 2b_{499} \\
&= b_{250} + b_{249} + 2b_{249} = b_{250} + 3b_{249} \\
&= b_{125} + b_{124} + 3b_{124} = b_{125} + 4b_{124} \\
&= b_{62} + 4(b_{62} + b_{61}) = 5b_{62} + 4b_{61} \\
&= 5(b_{31} + b_{30}) + 4b_{30} = 5b_{31} + 9b_{30} \\
&= 5b_{15} + 9(b_{15} + b_{14}) = 14b_{15} + 9b_{14} \\
&= 14b_7 + 9(b_7 + b_6) = 23b_7 + 9b_6 \\
&= 23b_3 + 9(b_3 + b_2) = 32b_3 + 9b_2 \\
&= 32b_1 + 9(b_1 + b_0) = 32 + 18 = 50.
\end{aligned}$$

Thus, there are 50 different representations of 2000 in this form. □

6. Let

$$f : (0, \infty) \to \mathbb{R}$$

be a function such that

(1) f is strictly increasing;
(2) $f(x) > -1/x$ for all $x > 0$;
(3) $f(x)f(f(x) + 1/x) = 1$ for all $x > 0$.

Find $f(1)$.

Solution. We first note that since $f(x) > -1/x$, we have

$$f(x) + 1/x > 0,$$

so $f(f(x) + 1/x)$ is in fact defined. Let

$$y = f(x) + 1/x.$$

Then,

$$f(x)f(y) = 1,$$

so $f(x) \neq 0$ for all $x > 0$ and $f(y) = 1/f(x)$. Also,

$$f(y)f(f(y) + 1/y) = 1,$$

so

$$\begin{aligned} f(x)f(y) &= f(y)f(f(y) + 1/y) \\ f(x) &= f\left(\frac{1}{f(x)} + \frac{1}{f(x) + 1/x}\right). \end{aligned}$$

Since $f(x)$ is strictly increasing, it is in particular one-to-one, so the values inside f must be equal. Thus, we have

$$\begin{aligned} x &= \frac{1}{f(x)} + \frac{1}{f(x) + 1/x} \\ xf(x)(f(x) + 1/x) &= f(x) + 1/x + f(x) \\ x^2 f^2(x) - xf(x) - 1 = 0. \end{aligned}$$

Letting $u = xf(x)$, we get

$$\begin{aligned} u^2 - u - 1 = 0 \\ u &= \frac{1 \pm \sqrt{1+4}}{2} \\ xf(x) &= \frac{1 \pm \sqrt{5}}{2} \\ f(x) &= \frac{1 \pm \sqrt{5}}{2x}. \end{aligned}$$

Suppose now that there exists an x such that $f(x) = \frac{1+\sqrt{5}}{2x}$. Suppose $y > x$. Then $f(y) = \frac{1 \pm \sqrt{5}}{2y}$, and in either case, $f(y) < f(x)$. Since $f(x)$ is strictly increasing, no such x could exist, so

$$f(x) = \frac{1 - \sqrt{5}}{2x}$$

for all $x > 0$. This function does in fact satisfy all three properties, so is the unique solution. Finally,

$$f(1) = \frac{1 - \sqrt{5}}{2}. \qquad \square$$