The Third Annual UNLV Mathematics Competition

Saturday, November 10th, 9:00 am to 12:30 pm. To receive full marks, solutions must be complete and well justified. Decisions of the judges are final. Solutions will be posted at
http://www.nevada.edu/~baragar/competitions.html

1. Let N be the be the set of all numbers in $\{1, 2, 3, ..., 1\,000\,000\}$ that can be written as the sum of a perfect square and a positive perfect cube. Let N^c be the rest of the numbers in $\{1, 2, 3, ..., 1\,000\,000\}$. Which set has more elements, N or N^c?

Solution. Let $n \in N$. Then there exist positive integers a and b such that

$$n = a^2 + b^3.$$

Since $n < 1\,000\,000$, we must have $a \leq 1000$ and $b \leq 100$, so there are at most (100)(1000) such n. That is, N has at most 100 000 elements, so N^c has at least 900 000 elements. Thus, N^c has more. ☐

2. A regular 2001-gon is divided into triangles by nonintersecting diagonals. How many of these triangles are acute? (An n-gon is *regular* if all sides and angles are congruent. A *diagonal* is a segment that joins two nonadjacent vertices. A triangle is *acute* if all of its angles are less than 90°.)

Solution. Note that, if a triangle is acute, then its circumcenter is inside the triangle. Since the 2001-gon is regular, all vertices lie on a single circle that is the circumcircle of all the created triangles. Since 2001 is odd, the center of this circle does not lie on any diagonal. Thus, it is on the interior of exactly one triangle, so there is exactly one acute triangle among all these triangles. ☐

3. Let $a_1 < a_2 < ... < a_{43} < a_{44}$ be positive integers not exceeding 125. Prove that, among the 43 differences $d_i = a_{i+1} - a_i$, $i = 1, 2, ..., 43$, some value must occur at least ten times.

Proof. Note that

$$\begin{aligned} a_{44} &= a_1 + (a_2 - a_1) + (a_3 - a_2) + ... + (a_{44} - a_{43}) \\ &= a_1 + d_1 + d_2 + ... + d_{43}. \end{aligned}$$

Note that $a_1 \geq 1$. Suppose no value is taken on ten times. Let us order these differences so that

$$1 \leq d_{i_1} \leq d_{i_2} \leq ... \leq d_{i_{43}}.$$

Then, since there are at most 9 differences equal to 1, we know $d_{i_{10}} \geq 2$. Similarly, $d_{i_{19}} \geq 3$, $d_{i_{28}} \geq 4$, and $d_{i_{37}} \geq 5$. Thus,

$$a_{44} \geq 1 + 9(1) + 9(2) + 9(3) + 9(4) + 7(5) = 1 + 9(10) + 7(5) = 126.$$

Since we were given that $a_{44} \leq 125$, we have a contradiction. Thus, at least one value must be taken on 10 times. □

4. Prove that, for any positive integer n,

$$2^{1/2}4^{1/4}8^{1/8} \cdots (2^n)^{1/2^n} < 4.$$

Proof. We note that

$$2^{1/2}4^{1/4}8^{1/8} \cdots (2^n)^{1/2^n} = 2^{\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+...+\frac{n}{2^n}}.$$

It is clear that

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + ... + \frac{n}{2^n} = \sum_{k=1}^{n} \frac{k}{2^k} < \sum_{k=1}^{\infty} \frac{k}{2^k}.$$

Thus, we wish to evaluate this infinite sum. There are a couple of ways of doing this. Our first method uses calculus. Recall that

$$(1) \qquad \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \qquad \text{for } |x| < 1.$$

Differentiating, we get

$$\sum_{k=1}^{\infty} kx^(k-1) = \frac{1}{(1-x)^2} \qquad \text{for } |x| < 1.$$

Multiplying by x and evaluating at $x = 1/2$, we get

$$\sum_{k=1}^{\infty} \frac{k}{2^k} = \frac{(1/2)}{(1/2)^2} = 2.$$

Thus,

$$2^{1/2}4^{1/4}8^{1/8} \cdots (2^n)^{1/2^n} < 2^{\sum_{k=1}^{\infty} \frac{k}{2^k}} = 2^2 = 4.$$

Our second way of evaluating this sum does not involve calculus. We know (1) (without using calculus) since it is the sum of the geometric series. Let

$$S = \sum_{k=1}^{\infty} \frac{k}{2^k}.$$

Then

$$\begin{aligned} S &= \sum_{k=1}^{\infty} \frac{1}{2^k} + \sum_{k=2}^{\infty} \frac{k-1}{2^k} \\ &= 2 + \frac{1}{2} \sum_{k=2}^{\infty} \frac{k-1}{2^{k-1}} \\ &= 2 + \frac{1}{2} S \\ \frac{1}{2} S &= 2 \\ S &= 4. \end{aligned}$$

Thus, the given sum is less than $S = 4$. □

5. Suppose f is a real positive continuous function on $\mathbb{R}$ with

$$\int_{-\infty}^{\infty} f(x) dx = 1.$$

Let $0 < \alpha < 1$ and suppose $[a, b]$ is an interval of minimal length with

$$\int_{a}^{b} f(x) dx = \alpha.$$

Prove that $f(a) = f(b)$.

Proof. Consider the equation

$$F(x) = \int_{0}^{x} f(t) dt.$$

Since $f(t)$ is continuous, $F(x)$ is differentiable with $F'(x) = f(x)$. Let us define y implicitly as a function of x by

$$F(y + x) - F(x) = \alpha.$$

Then y is the length of the interval beginning at x with the property that

$$\int_{x}^{x+y} f(t) dt = \alpha.$$

We are given that y is minimal when $x = a$ and $y = b - a$. Using implicit differentiation,

$$\begin{aligned} F'(y + x)(y' + 1) - F'(x) &= 0 \\ f(y + x)(y' + 1) - f(x) &= 0. \end{aligned}$$

Since y is minimal when $x = a$, $y' = 0$ when $x = a$, so

$$f(b) - f(a) = 0.$$

That is, $f(a) = f(b)$. ☐

6. Let $\mathbb{R}_{\text{finite}}$ be the set of all finite subsets of $\mathbb{R}$. Suppose

$$f : \mathbb{R} \to \mathbb{R}_{\text{finite}}.$$

Prove that there are x, $y \in \mathbb{R}$ such that $x \notin f(y)$ and $y \notin f(x)$.

Proof. Consider the set

$$S = \bigcup_{n=1}^{\infty} f(n).$$

Since this is a countable union of finite sets, S is countable. Since $\mathbb{R}$ is uncountable, there exists a $y \notin S$. Since $f(y)$ is finite, there exists a positive integer x that is not in $f(y)$. Since $f(x) \subset S$ and $y \notin S$, we have $y \notin f(x)$. Thus, we have found x and y such that $x \notin f(y)$ and $y \notin f(x)$. ☐