MISCELLANEOUS EXERCISES FROM THE 2002 IMO WINTER CAMP

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Exercise 1, Putnam, '98, B2. Let $P = (a, b)$ be a fixed point with $0 < b < a$. Let Q be a point on $y = x$ and R a point on $y = 0$. Find the smallest possible value for the perimeter of ΔPQR as a function of a and b.

Exercise 2, Shortlist, IMO '88. Let α be the largest root of $x^3 - 3x^2 + 1$. Show 17 divides $\lfloor \alpha^{1788} \rfloor$ and $\lfloor \alpha^{1988} \rfloor$.

Exercise 3, Putnam, '97, A1. Let ΔABC have orthocenter H and circumcenter O. Let A' be the midpoint of BC and let D be the base of the altitude at A. Suppose $A'OHD$ is a rectangle with $|OH| = 11$ and $|OA'| = 5$. What is $|BC|$?

Exercise 4, IMO, '74, # 5. Show 5 does not divide

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}.$$

Exercise 5. A regular 1997-gon is decomposed into triangles using non-intersecting diagonals. How many of these triangles are acute?

Exercise 6. Let

$$f : (0, \infty) \to \mathbb{R}.$$

Suppose f is increasing,

$$f(x) > \frac{-1}{x},$$

and

$$f(x) f\left(f(x) + \frac{1}{x}\right) = 1$$

for all x in the domain of f. What is $f(1)$?

Pedal triangles: Given a triangle ΔABC and a point P, the perpendiculars through P to the sides BC, AC, and AB, intersect those sides at X, Y, and Z, respectively. The triangle ΔXYZ is called the *pedal triangle* for P with respect to ΔABC. The next few questions refer to pedal triangles.

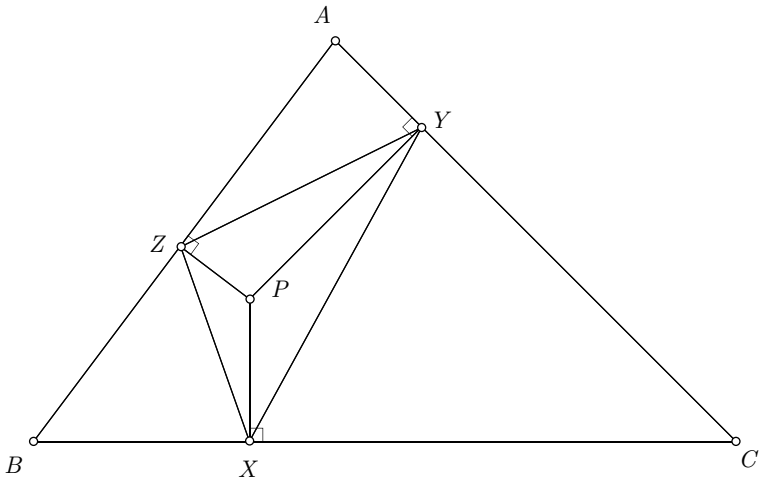

FIGURE 1.

Exercise 7. Show

$$|YZ| = |AP| \sin A.$$

Exercise 8. Show

$$\angle APB = \angle C + \angle XZY.$$

(This isn't always true – state and prove what is true in general.)

Exercise 9, The Simson Line. Prove that X, Y, and Z are collinear if and only if P is on the circumcircle of ΔABC.

Exercise 10. In how many ways can 5 identical balls be arranged in 10 bowls (labelled 1 to 10). Generalize to N (identical or numbered) balls in M (numbered or identical) bowls.

Exercise 11. How many sequences $\{a, b, c, d, e\}$ are there satisfying $0 \leq a \leq b \leq c \leq d \leq e \leq 9$? Generalize to sequences of N non-negative non-decreasing integers less than or equal to M.