Mock Olympiad

Friday, January 4th, 3 1/2 hours

Exercise 1, Putnam, '01, B1. Let n be an even positive integer. Write the numbers 1, 2,..., n^2 in the squares of an $n \times n$ grid so that the k-th row, from left to right, is

$$(k-1)n+1, (k-1)n+2, ..., (k-1)n+n.$$

Color the squares of the grid so that half of the squares in each row and each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

Exercise 2. Let b_n be the number of ways that n can be written in the form

$$n = a_0 + a_1 2 + a_2 2^2 + ... + a_m 2^m,$$

where m is arbitrary, $a_k \in \{0, 1, 2\}$, and $a_m \neq 0$. What is b_{2002}?

Exercise 3. Let Γ be the inscribed circle of a triangle ΔABC. Let the parallel to BC that is tangent to Γ intersect AB and AC at D and E. Let $|BC| = a$ and $|DE| = b$. Find the perimeter of ΔABC in terms of a and b.

Exercise 4. Let

$$f(x) = x^n + a_{n-1}x^{n-1} + ... + a_1 x + 1$$

have non-negative coefficients and n real roots. Prove

$$f(2) \geq 3^n.$$

Exercise 5. Suppose a_0, a_1,... is a sequence of positive real numbers such that $a_0 = 1$ and

$$a_n = a_{n+1} + a_{n+2}$$

for $n \geq 0$. Find a_n.

Exercise 6. Let A' be the midpoint of BC in ΔABC. Let P be a point on AA'. Extend BP to meet AC at E, and extend CP to meet AB at F. If the incircles of ΔBPF and ΔCPE have the same radius, prove that ΔABC is isosceles.