

Mock Olympiad

Monday, January 7th, 3 1/2 hours

Exercise 1. Let $P(x)$ be a polynomial of degree at most n and such that

$$P(k) = 2^k$$

for $k = 0, 1, 2, \dots, n$. Find $P(n+1)$.

Exercise 2, Japan, 2001, # 1. An $m \times n$ checkerboard is painted red and black with the following property: For every black square, the number of adjacent (share an edge) black squares is odd. Prove that the number of black squares is even.

Exercise 3, Turkey, '97, # 2 (modified). Let $a_1 = \alpha$ and $b_1 = \beta$. Define

$$\begin{aligned} a_{n+1} &= \alpha a_n - \beta b_n \\ b_{n+1} &= \beta a_n + \alpha b_n. \end{aligned}$$

Find the number of real pairs (α, β) such that

$$a_{1997} = a_1 \quad \text{and} \quad b_{1997} = b_1.$$

Exercise 4, Moldova, '00, # 12.8. The quadrilateral $ABCD$ circumscribes a circle of radius r . The points of tangency are E, F, G , and H , on AB, BC, CD , and DA , respectively. Let r_1, r_2, r_3 , and r_4 be the radii of $\triangle EBF, \triangle FCG, \triangle GDH$, and $\triangle HAE$, respectively. Prove

$$r_1 + r_2 + r_3 + r_4 \geq 2(2 - \sqrt{2})r.$$

Exercise 5, United Kingdom, '98, # 8. Let x, y , and z be positive integers such that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$$

and $\gcd(x, y, z) = 1$. Prove that both $x - z$ and xyz are perfect squares.

Exercise 6, Ukraine, '01, # 8. Let a, b, c, α, β , and γ be positive real numbers such that $\alpha + \beta + \gamma = 1$. Prove

$$\alpha a + \beta b + \gamma c + 2\sqrt{(\alpha\beta + \beta\gamma + \gamma\alpha)(ab + bc + ca)} \leq a + b + c.$$

Exercise 7, Taiwan, '00, # 2. In an acute $\triangle ABC$ with $|AC| > |BC|$, let C' be the midpoint of AB , AD the altitude at A , BE the altitude at B , and H the orthocenter. Let AB and DE intersect at R . Prove RH and CC' are perpendicular.