THE ASYMPTOTIC GROWTH OF INTEGER SOLUTIONS TO THE ROSENBERGER EQUATIONS

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Abstract. Zagier showed that the number of integer solutions to the Markoff equation with components bounded by T grows asymptotically like $C(\log T)^2$, where C is explicitly computable. Rosenberger showed that there are only a finite number of equations $ax^2 + by^2 + cz^2 = dxyz$ with a, b, and c dividing d, and for which the equation admits an infinite number of integer solutions. In this paper, we generalize Zagier's techniques so that they may be applied to the Rosenberger equations. We also apply these techniques to the equations $ax^2 + by^2 + cz^2 = dxyz + 1$.

Introduction

The Markoff equation

$$x^2 + y^2 + z^2 = 3xyz \tag{1}$$

was studied by Markoff (1879) [Ma], who demonstrated a relationship between its integer solutions and Diophantine approximation. The equation is also interesting as a Diophantine equation. Its set of integer solutions is infinite and nontrivial, yet is easy to describe. The Markoff equation is quadratic in each variable, so given a solution (x, y, z), we can find a new solution $(3yz - x, y, z)$. Using this map, permutations of the variables, and the *fundamental solution* $(1, 1, 1)$, we can construct the *Markoff tree* $\mathfrak{M}$ of positive ordered solutions, shown in Figure 1. Every nontrivial integer solution to Eq. 1 is derived from a solution in this tree by applying a permutation of the variables and sign changes in pairs (see, for example, [C]). The trivial solution is $(0, 0, 0)$.

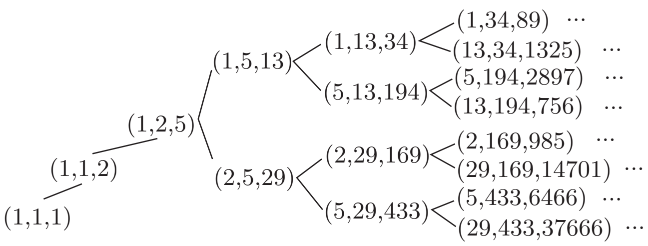

Figure 1. The Markoff tree $\mathfrak{M}$.

Zagier [Z] considered the quantity

$$N(T) = \#\{(x, y, z) \in \mathfrak{M} : z < T\}$$

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and proved

$$N(T) = C(\log T)^2 + O(\log T(\log\log T)^2),$$

where C is explicitly computable and $C \approx .180717104712$. Note that there is a typo in [Z] – the seventh digit of C is omitted.

There are several generalizations of the Markoff equation, including the Hurwitz equations,

$$x_1^2 + ... + x_n^2 = ax_1 \cdots x_n, \tag{2}$$

investigated by Hurwitz [Hu]; equations studied by Mordell [Mo],

$$x^2 + y^2 + z^2 = axyz + b; \tag{3}$$

variations studied by Rosenberger [R],

$$ax^2 + by^2 + cz^2 = dxyz; \tag{4}$$

and a variation studied by Jin and Schmidt [J-S],

$$ax^2 + by^2 + cz^2 = dxyz + 1. \tag{5}$$

In these last two classes of equations, we further require that a, b, and c divide d.

Zagier's techniques are not applicable to the Hurwitz equations, Eq. 2 (see [B1]). The application to those equations studied by Mordell (Eq. 3) is straight forward when it applies [B2]. In this paper, we will generalize Zagier's techniques so that they may be applied to the Rosenberger variations, Eq. 4. The application is not straight forward, since we will not be able to exploit the symmetry of the equation.

Every equation of the form Eq. 4 has the trivial solution $(0, 0, 0)$. Rosenberger showed that if such an equation includes a nontrivial integer solution and $a \le b \le c$, then the equation is one of six equations. These six equations include the Markoff equation Eq. 1, and

$$\begin{aligned} R_1 : &\quad x^2 + y^2 + 2z^2 = 4xyz \\ R_2 : &\quad x^2 + 2y^2 + 3z^2 = 6xyz \\ R_3 : &\quad x^2 + y^2 + 5z^2 = 5xyz. \end{aligned}$$

The last two equations are

$$\begin{aligned} R_4 : &\quad x^2 + y^2 + z^2 = xyz \\ R_5 : &\quad x^2 + y^2 + 2z^2 = 2xyz. \end{aligned}$$

An integer triple (x, y, z) is a solution to the Markoff equation if and only if $(3x, 3y, 3z)$ is a solution to R_4. Thus, the Markoff equation and R_4 are essentially the same. Similarly, an integer triple (x, y, z) is a solution to R_1 if and only if $(2x, 2y, 2z)$ is a solution to R_5.

Let

$$H(x, y, z) = |x| + |y| + |z|$$

be a height on integer triples. Let

$$N_m(T) = \#\{(x, y, z) \in \mathbb{Z}^3 : (x, y, z) \text{ is a solution to } R_m \text{ and } H(x, y, z) < T\}.$$

Our main result is the following:

Theorem 0.1. *The number of integer solutions to the Rosenberger equation R_m with height bounded by T grows asymptotically like*

$$N_m(T) = C_m \log^2 T + O(\log T(\log\log T)^2),$$

where $C_1 \approx 1.63142834189$, $C_2 \approx 1.66271739346$, and $C_3 \approx 3.52831194430$.

We also have $C_4 = 18C \approx 3.25290788481$, where C is the constant found by Zagier, and $C_5 = C_1$.

In Section 5, we apply the technique to Eq. 5, though we leave checking many details to the reader. We also discuss what portions of our results are applicable to equations of the form

$$ax^2 + by^2 + cz^2 = dxyz + e,$$

where the coefficients are integers, and a, b, and c divide d.

1. The Rosenberger variations

Let us write Eq. 4 in the following fashion:

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = dx_1x_2x_3. \tag{6}$$

From a solution $\mathbf{x} = (x_1, x_2, x_3)$ to Eq. 6, we can generate three new solutions by applying the automorphisms:

$$\begin{aligned}
\phi_1(\mathbf{x}) &= \left(\frac{d}{a_1}x_2x_3 - x_1, x_2, x_3\right)\\
\phi_2(\mathbf{x}) &= \left(x_1, \frac{d}{a_2}x_1x_3 - x_2, x_3\right)\\
\phi_3(\mathbf{x}) &= \left(x_1, x_2, \frac{d}{a_3}x_1x_2 - x_3\right).
\end{aligned}$$

We will call a solution $\mathbf{x}$ a *positive integral solution* if all the components of $\mathbf{x}$ are positive integers. If $\mathbf{x}$ is a positive integral solution, then so is $\phi_i(\mathbf{x})$. One can see this by noting that the product of x_i and $\frac{d}{a_i}x_jx_k - x_i$ is $a_jx_j^2 + a_kx_k^2$, which is clearly positive. If $x_i = 0$ for any i, then $\mathbf{x} = (0, 0, 0)$. Thus, every nontrivial integer solution can be obtained from a positive integer solution by a couple of sign changes.

Note that we cannot have both $H(\phi_i(\mathbf{x})) \le H(\mathbf{x})$ and $H(\phi_j(\mathbf{x})) \le H(\mathbf{x})$, for then we would have

$$\begin{aligned}
\frac{d}{a_i}x_jx_k &\le x_i\\
\frac{d}{a_j}x_ix_k &\le x_j
\end{aligned}$$

so

$$\begin{aligned}
dx_ix_jx_k &\le 2a_ix_i^2\\
dx_ix_jx_k &\le 2a_jx_j^2\\
dx_ix_jx_k &\le a_ix_i^2 + a_jx_j^2\\
&< a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = dx_1x_2x_3.
\end{aligned}$$

Thus, descent, when it occurs, is unique.

Since descent is unique and cannot continue indefinitely, there exists a *fundamental solution* $\mathbf{w}$ from which we cannot descend. By investigating the properties of fundamental solutions, Rosenberger concluded that there are only six equations of this form that have an infinite set of integer solutions. Both of the equations R_1 and R_2 have the single fundamental solution $(1,1,1)$, and R_3 has the two fundamental solutions $(1,2,1)$ and $(2,1,1)$. Note that, for each of the fundamental solutions $\mathbf{w}$ for equations R_1, R_2, and R_3, we have $H(\phi_3(\mathbf{w})) = H(\mathbf{w})$ and $H(\phi_j(\mathbf{w})) > H(\mathbf{w})$ for $j = 1$ and 2.

We will study the integer solutions to these Rosenberger variations by studying the *tree of solutions* $\mathfrak{T}_{\mathbf{y}}$ for a positive integer solution $\mathbf{y}$. This tree is rooted at $\mathbf{y}$ and is generated as follows. For any $\mathbf{x}$ in $\mathfrak{T}_{\mathbf{y}}$, there exists a permutation i, j, k of $\{1,2,3\}$ such that $H(\phi_i(\mathbf{x})) \leq H(\mathbf{x})$. The *daughters* of $\mathbf{x}$ are the nodes $\phi_j(\mathbf{x})$ and $\phi_k(\mathbf{x})$. The tree $\mathfrak{T}_{(1,1,1)}$ for R_1 is shown in Figure 2. This tree contains every positive integer solution to R_1.

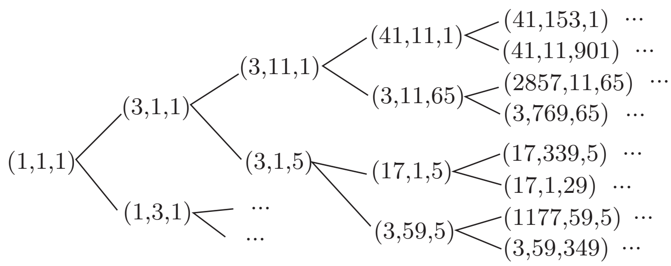

FIGURE 2. The tree of positive integer solutions to the equation $x^2 + y^2 + 2z^2 = 4xyz$.

2. Zagier's Argument

In this section, we condense and generalize Zagier's techniques. There are three main ideas. The first is to compare the tree $\mathfrak{T}_{\mathbf{y}}$ with the Euclid tree. The *Euclid tree* $\mathfrak{E}$ is the tree rooted at $(1,1)$ and generated by the branching operations $\sigma_1(a,b) = (a, a+b)$ and $\sigma_2(a,b) = (b, a+b)$. This tree contains all ordered coprime pairs twice and going down the tree is the Euclidean algorithm, hence the tree's name. We will be interested in Euclid trees $\mathfrak{E}_{(\alpha,\beta)}$ rooted at an arbitrary pair (α,β). Let

$$E_{(\alpha,\beta)}(t) = \#\{(a,b) \in \mathfrak{E}_{(\alpha,\beta)} : a+b < t\}.$$

It is well known that $E_{(1,1)}(t)$ grows asymptotically like $\frac{3}{\pi^2}t^2$ (see, for example, [H-W, p266]). More generally, for $\beta \geq \alpha > 0$,

$$E_{(\alpha,\beta)}(t) = \frac{3t^2}{\pi^2\alpha\beta} + O\left(\frac{t\log t}{\alpha}\right),$$

as is shown in [Z]. (Zagier's error term is slightly better than this, but this is enough for our argument.)

To compare the trees, we define a map Ψ from the tree $\mathfrak{T}_{\mathbf{y}}$ to the tree $\mathfrak{E}_{(\alpha,\beta)}$. Our definition is inductive. For each $\mathbf{x} \in \mathfrak{T}_{\mathbf{y}}$, there exists an $i \in \{1,2,3\}$ such that $H(\phi_i(\mathbf{x})) \leq H(\mathbf{x})$. For $\mathbf{x} = \mathbf{y}$, we fix j and k. We let $\Psi(\mathbf{y}) = (\alpha,\beta)$, $\Psi(\phi_j(\mathbf{y})) = (\beta, \alpha+\beta)$ and $\Psi(\phi_k(\mathbf{y})) = (\alpha, \alpha+\beta)$. We define the rest of Ψ inductively. For all other $\mathbf{x} \in \mathfrak{T}_{\mathbf{y}}$, there exists a permutation (i,j,k) of $(1,2,3)$ such that $H(\phi_i(\mathbf{x})) <$

$H(\mathbf{x})$ and $H(\phi_j\phi_i(\mathbf{x})) \leq H(\phi_i(\mathbf{x}))$. (This choice almost always gives $x_i \geq x_j \geq x_k$, but there are some exceptions.) If $\Psi(\mathbf{x}) = \mathbf{s}$, then let

$$\Psi(\phi_j(\mathbf{x})) = (s_2, s_1 + s_2)$$
$$\Psi(\phi_k(\mathbf{x})) = (s_1, s_1 + s_2).$$

The map Ψ from $\mathfrak{T}_{(1,1,1)}$ for R_2 to the tree $\mathfrak{E}_{(1,1)}$ is shown in Figure 3.

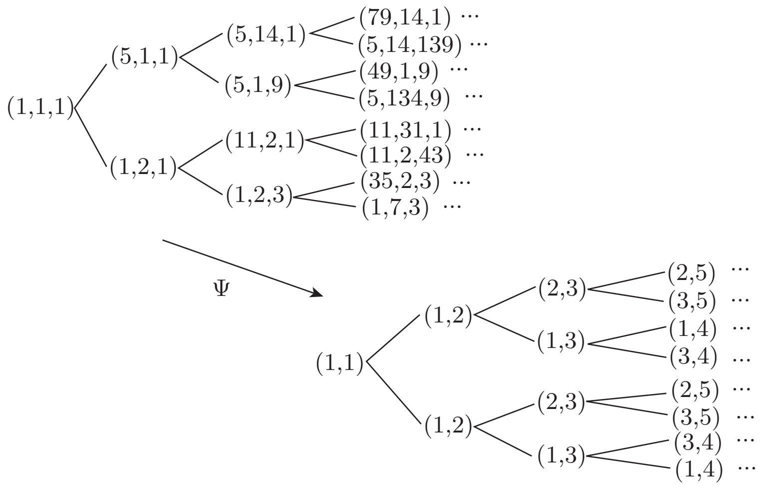

FIGURE 3. The tree $\mathfrak{T}_{(1,1,1)}$ for the equation $x_1^2 + 2x_2^2 + 3x_3^2 = 6x_1x_2x_3$, and the map Ψ to the Euclid tree $\mathfrak{E}_{(1,1)}$.

The second main idea is an averaging technique. Averaging techniques are fairly common – for example, Tate used the idea when defining canonical heights on elliptic curves [S, p 228]. If we fix a solution $\mathbf{p} = (p_1, p_2, p_3)$ to Eq. 6, then the branch of the tree $\mathfrak{T}_{\mathbf{p}}$ with $x_1 = p_1$ is given by alternately applying ϕ_2 and ϕ_3. The composition $\phi_2\phi_3$ generates a linear action on (x_2, x_3), which has the eigenvalue

$$\lambda_{p_1} = \left(\frac{m_1 p_1 + \sqrt{m_1^2 p_1^2 - 4}}{2} \right)^2$$

and its multiplicative inverse, where $m_1 = \frac{d}{\sqrt{a_2 a_3}}$. If this eigenvalue is not one, then in the long run, the action looks like multiplication by this eigenvalue. Taking a cue from Zagier, we therefore define

$$f_i(x) = \log \left(\frac{m_i x + \sqrt{m_i^2 x^2 - 4}}{2} \right),$$

where $m_i = \frac{d}{\sqrt{a_j a_k}}$.

Suppose now that $\mathbf{x}$ is a solution to Eq. 6 with $H(\phi_i(\mathbf{x})) \leq H(\mathbf{x})$. Let

$$x_{i*} = f_i^{-1}(f_j(x_j) + f_k(x_k)).$$

(Note that f_i^{-1} exists, since f_i is an increasing function in x.) Let $\mathbf{x}_*$ be the vector obtained from $\mathbf{x}$ by substituting x_i with x_{i*}. Then $\mathbf{x}_*$ satisfies the equation

$$a_i x_{i*}^2 + a_j x_j^2 + a_k x_k^2 = d x_{i*} x_j x_k + \frac{4a_1 a_2 a_3}{d^2}. \tag{7}$$

To see this, let us first let $u_i = e^{f_i(x_{i*})}$, $u_j = e^{f_j(x_j)} = \sqrt{\lambda_{x_j}}$, and $u_k = e^{f_k(x_k)} = \sqrt{\lambda_{x_k}}$. Then $u_i = u_j u_k$. We also note that u_i is a root of the quadratic $t^2 - m_i x_{i*} t + 1$, so the other root is u_i^{-1} and the sum of the roots is $u_i + u_i^{-1} = m_i x_{i*}$. Similarly, $m_j x_j = u_j + u_j^{-1}$ and $m_k x_k = u_k + u_k^{-1}$. Plugging these expressions for x_{i*}, x_j and x_k into the expression

$$a_i x_{i*}^2 + a_j x_j^2 + a_k x_k^2 - d x_{i*} x_j x_k,$$

and simplifying, we get $\frac{4a_1a_2a_3}{d^2}$. Thus, the map $\mathbf{x} \to (f_k(x_k), f_j(x_j))$ is a good approximation of the map Ψ. Let us make this last statement more precise.

Lemma 2.1. *For any $\mathbf{x} \in \mathfrak{T}_{\mathbf{y}}$, let (i,j,k) be a permutation of $\{1,2,3\}$ such that $H(\phi_i(\mathbf{x})) \le H(\mathbf{x})$. Suppose there exist $r_i < 2a_i$ such that $r_i x_i > d x_j x_k$. Then*

$$x_i < x_{i*} < x_i + \frac{K_i}{x_i}.$$

The constant K_i depends on i and the constants a_1, a_2, a_3, d, and r_i.

Proof. First, note that $x_{i*} > x_i$. To see this, think of x_i and x_{i*} as roots of the appropriate parabolas suggested by Equations 6 and 7. The shapes of these parabolas are identical, but the parabola for x_{i*} is shifted down. Thus, the roots x_i and x_i' of Eq. 6 are between the roots x_{i*} and x_{i*}' of Eq. 7. Since x_i and x_{i*} are the larger roots of their respective equations, we get $x_i < x_{i*}$.

Let us now take the difference of Equations 6 and 7. This gives

$$\begin{aligned} a_i x_{i*}^2 - a_i x_i - d x_{i*} x_j x_k + d x_i x_j x_k &= \frac{4a_1a_2a_3}{d^2} \\ (x_{i*} - x_i)(a_i x_{i*} + a_i x_i - d x_j x_k) &= \frac{4a_1a_2a_3}{d^2} \\ (x_{i*} - x_i)(2a_i - r_i) x_i &< \frac{4a_1a_2a_3}{d^2} \\ x_i^* &< x_i + \frac{4a_1a_2a_3}{d^2(2a_i - r_i)x_i} \\ &< x_i + \frac{K_i}{x_i}. \end{aligned}$$

□

Theorem 2.2. *Suppose $\mathbf{w}$ is a fundamental solution and that $f_m(w_m) > 0$ for $m = 1$, 2, and 3. For each $\mathbf{x} \in \mathfrak{T}_{\mathbf{w}}$, let (i,j,k) be a permutation of $\{1,2,3\}$ such that $H(\phi_i(\mathbf{x})) \le H(\mathbf{x})$. Suppose there exist $r_m < 2a_m$ such that, for all but finitely many $\mathbf{x} \in \mathfrak{T}_{\mathbf{w}}$, we have $r_i x_i > x_j x_k$. Then, for all but finitely many $\mathbf{y} \in \mathfrak{T}_{\mathbf{w}}$, we have*

$$N_{\mathbf{y}}(T) = \frac{3\log^2 T}{\pi^2 f_j(y_j) f_k(y_k)} + O\left(\frac{\log^2 T}{y_i^2 f_j(y_j) f_k^2(y_k)}\right) + O\left(\frac{\log T \log\log T}{f_k(y_k)}\right),$$

where $H(\phi_i(\mathbf{y})) \le H(\mathbf{y})$ and, if $\mathbf{y} \ne \mathbf{w}$, $H(\phi_j\phi_i(\mathbf{y})) \le H(\phi_i(\mathbf{y}))$.

Proof. Let us first note that for any $\mathbf{x} \in \mathfrak{T}_{\mathbf{y}}$ with $H(\phi_i(\mathbf{x})) \le H(\mathbf{x})$, we have

$$\begin{aligned} x_i &\ge \frac{d}{a_i} x_j x_k - x_i \\ 2a_i x_i &\ge d x_j x_k. \end{aligned}$$

In particular, x_j, $x_k \le 2a_i x_i$.

Let $(\alpha, \beta) = (f_k(y_k), f_j(y_j))$. Let Ψ be the map from the tree $\mathfrak{T}_{\mathbf{y}}$ to the tree $\mathfrak{E}_{(\alpha,\beta)}$. Let $\mathbf{x} \in \mathfrak{T}_{\mathbf{y}}$ and suppose $\Psi(\mathbf{x}) = (s_1, s_2)$. We claim that $f_k(x_k) \leq s_1$ and $f_j(x_j) \leq s_2$, which we prove using induction. As a consequence of our choice of (α, β), it is true for the base case. Suppose it is true for $\mathbf{x}$. Then, by Lemma 2.1,

$$f_i(x_i) \leq f_i(x_{i*}) = f_j(x_j) + f_k(x_k) \leq s_1 + s_2.$$

Thus, the inequalities are true for the two daughters of $\mathbf{x}$, which completes the induction. Now, suppose we order a_1, a_2, and a_3 so that $a_1 \leq a_2 \leq a_3$. Then $f_1(x) \leq f_2(x) \leq f_3(x)$. Suppose

$$s_1 + s_2 < f_1\left(\frac{T}{6a_3}\right).$$

Then

$$\begin{aligned} f_i(x_i) &\leq s_1 + s_2 < f_1\left(\frac{T}{6a_3}\right) \leq f_i\left(\frac{T}{6a_3}\right) \\ x_i &< \frac{T}{6a_3} \\ H(\mathbf{x}) &< T, \end{aligned}$$

where in the last, we used that x_j and $x_k \leq 2a_i x_i \leq 2a_3 x_i$. Thus,

$$N_{\mathbf{y}}(T) \geq E_{(\alpha,\beta)}(f_1(T/6a_3)) = E_{(\alpha,\beta)}(\log(T) + O(1)). \tag{8}$$

By Lemma 2.1,

$$f_j(x_j) + f_k(x_k) = f_i(x_{i*}) < f_i\left(x_i + O\left(\frac{1}{x_i}\right)\right) = f_i(x_i) + O\left(\frac{1}{x_i^2}\right). \tag{9}$$

Let

$$(\alpha', \beta') = \left(f_k(y_k) - \frac{c}{y_i^2}, f_j(y_j) - \frac{c}{y_i^2}\right), \tag{10}$$

where $\frac{c}{4a_3^2}$ is a constant that bounds the functions implied by the big O in Eq. 9 for each of $i = 1$, 2, and 3. If y_i is large enough, then α', $\beta' > 0$. Since y_j, $y_k < 2a_3 y_i$, we have that y_i is large enough for all but finitely many $\mathbf{y}$ in $\mathfrak{T}_{\mathbf{w}}$.

Since we will choose permutations of $\{1, 2, 3\}$ for each $\mathbf{x} \in \mathfrak{T}_{\mathbf{y}}$, let us fix $i(\mathbf{y})$ so that $H(\phi_{i(\mathbf{y})}(\mathbf{y})) \leq H(\mathbf{y})$. Note that $x_{i(\mathbf{y})} \geq y_{i(\mathbf{y})}$ for all $\mathbf{x} \in \mathfrak{T}_{\mathbf{y}}$. Suppose $\Psi(\mathbf{x}) = \mathbf{s}$. We claim that $f_k(x_k) - \frac{c}{y_{i(\mathbf{y})}^2} \geq s_1$ and $f_j(x_j) - \frac{c}{y_{i(\mathbf{y})}^2} \geq s_2$, which we again prove using induction. Our choice of α', β' covers the base case. Suppose it is true for $\mathbf{x}$. Then

$$f_i(x_{i*}) \leq f_i(x_i) + \frac{c}{4a_3^2 x_i^2}. \tag{11}$$

Since $x_{i(\mathbf{y})} \leq 2a_3 x_i$, we get

$$f_i(x_{i*}) \leq f_i(x_i) + \frac{c}{y_{i(\mathbf{y})}^2}.$$

Thus

$$f_i(x_i) - \frac{c}{y_{i(\mathbf{y})}^2} \geq f_i(x_{i*}) - \frac{2c}{y_{i(\mathbf{y})}^2} = f_j(x_j) + f_k(x_k) - \frac{2c}{y_{i(\mathbf{y})}^2} \geq s_1 + s_2.$$

This completes the induction step. Now, suppose $H(\mathbf{x}) < T$. Then

$$\begin{aligned} x_i &< T \\ f_i(x_i) &< f_i(T) = \log(T) + O(1) \\ s_1 + s_2 &< \log(T) + O(1). \end{aligned}$$

Thus,

$$E_{(\alpha',\beta')}(\log T + O(1)) \geq N_{\mathbf{y}}(T).$$

Combining this with Eq. 8, we get

$$N_{\mathbf{y}}(T) = \frac{3\log^2 T}{\pi^2(f_j(y_j) + O(1/y_i^2))(f_k(y_k) + O(1/y_i^2))} + O\left(\frac{\log T \log\log T}{\min\{f_k(y_k), f_j(y_j)\}}\right).$$

In most cases, we expect $\min\{f_k(y_k), f_j(y_j)\} = f_k(y_k)$. In the rare cases that this is not the case, we have $y_k \leq 2a_j y_j$, so

$$O(\min\{f_k(y_k), f_j(y_j)\}) = O(f_k(y_k)).$$

Thus, we get

$$N_{\mathbf{y}}(T) = \frac{3\log^2 T}{\pi^2 f_j(y_j) f_k(y_k)} + O\left(\frac{\log^2 T}{y_i^2 f_j(y_j) f_k^2(y_k)}\right) + O\left(\frac{\log T \log\log T}{f_k(y_k)}\right),$$

as claimed. □

This theorem is not particularly useful when $\mathbf{y}$ is small, since the first error term dominates. However, we do immediately get the following result, which will be useful later on.

Corollary 2.3. *Suppose* $\mathbf{w}$ *is a fundamental solution,* $f_m(w_m) > 0$ *for* $m = 1$, 2, *and* 3, *and* $\mathbf{y} \in \mathfrak{T}_{\mathbf{w}}$. *Then*

$$N_{\mathbf{y}}(T) = O(\log^2 T).$$

Furthermore, the constant implied by the big O can be chosen so that it is independent of $\mathbf{y}$ *(though it depends on* $\mathbf{w}$*).*

For $\mathbf{y}$ very large, the approximation in Theorem 2.2 is very good. Zagier's third main idea exploits that feature. Let $\mathfrak{T}_{\mathbf{y}}(U)$ be the subtree of $\mathfrak{T}_{\mathbf{y}}$ that includes all $\mathbf{x} \in \mathfrak{T}_{\mathbf{y}}$ such that $H(\mathbf{x}) \leq U$. The *boundary of* $\mathfrak{T}_{\mathbf{y}}(U)$ is the set $\partial\mathfrak{T}_{\mathbf{y}}(U)$ of solutions $\mathbf{x}$ with $H(\mathbf{x}) > U$ and $\phi_i(\mathbf{x}) \in \mathfrak{T}_{\mathbf{y}}(U)$ for some i. Then, for $U < T$, we can write

$$N_{\mathbf{y}}(T) = N_{\mathbf{y}}(U) + \sum_{\mathbf{x}\in\partial\mathfrak{T}_{\mathbf{y}}(U)} N_{\mathbf{x}}(T). \tag{12}$$

We estimate this using Theorem 2.2 and its corollary. The details are in the next theorem, which is the main result of this section.

Theorem 2.4. *Suppose* $\mathbf{w}$ *is a fundamental solution and that* $f_m(w_m) > 0$ *for* $m = 1$, 2, *and* 3. *For each* $\mathbf{x} \in T_{\mathbf{w}}$, $\mathbf{x} \neq \mathbf{w}$, *let* (i,j,k) *be a permutation of* $\{1,2,3\}$ *such that* $H(\phi_i(\mathbf{x})) \leq H(\mathbf{x})$ *and* $H(\phi_j\phi_i(\mathbf{x})) \leq H(\phi_i(\mathbf{x}))$. *For* $\mathbf{x} = \mathbf{w}$, *let* $(i,j,k) = (3,2,1)$. *Suppose there exist* $r_m < 2a_m$ *such that, for all but finitely many* $\mathbf{x} \in \mathfrak{T}_{\mathbf{w}}$, *we have* $r_i x_i > x_j x_k$. *Then for any* $\mathbf{y} \in \mathfrak{T}_{\mathbf{w}}$, *the constant*

$$C_{\mathbf{y}} = \frac{3}{\pi^2}\frac{1}{f_j(y_j)f_k(y_k)} + \lim_{U\to\infty}\frac{3}{\pi^2}\sum_{\mathbf{x}\in\mathfrak{T}_{\mathbf{y}}(U)} \frac{f_j(x_j) + f_k(x_k) - f_i(x_i)}{f_i(x_i)f_j(x_j)f_k(x_k)}$$

exists and

$$N_{\mathbf{y}}(T) = C_{\mathbf{y}}\log^2 T + O(\log T(\log\log T)^2).$$

Proof. Let us use Theorem 2.2 and its corollary to expand Eq. 12 as

$$N_{\mathbf{y}}(T) = O(\log^2 U) + C_U \log^2 T + O(D_U \log^2 T + E_U \log T \log\log T), \tag{13}$$

where

$$\begin{aligned} C_U &= \frac{3}{\pi^2} \sum_{\mathbf{x}\in\partial\mathfrak{T}_{\mathbf{y}}(U)} \frac{1}{f_j(x_j) f_k(x_k)} \\ D_U &= \sum_{\mathbf{x}\in\partial\mathfrak{T}_{\mathbf{y}}(U)} \frac{1}{x_i^2 f_j(x_j) f_k^2(x_k)} \\ E_U &= \sum_{\mathbf{x}\in\partial\mathfrak{T}_{\mathbf{y}}(U)} \frac{1}{f_k(x_k)}. \end{aligned} \tag{14}$$

Let $g(x_j, x_k)$ be an arbitrary function on $\mathfrak{T}_{\mathbf{y}}$ and let $\mathfrak{T}'_{\mathbf{y}}$ be a finite subtree of $\mathfrak{T}_{\mathbf{y}}$ that contains $\mathbf{y}$ and is connected. Then

$$\sum_{\mathbf{x}\in\partial\mathfrak{T}'_{\mathbf{y}}} g(x_j, x_k) = g(y_j, y_k) + \sum_{\mathbf{x}\in\mathfrak{T}'_{\mathbf{y}}} \left(g(x_i, x_j) + g(x_i, x_k) - g(x_j, x_k)\right). \tag{15}$$

This result, which may be thought of as a version of Green's theorem, is easily proved using induction. Using Eq. 15, we have

$$C_U = \frac{3}{\pi^2 f_j(y_j) f_k(y_k)} + \frac{3}{\pi^2} \sum_{\mathbf{x}\in\mathfrak{T}_{\mathbf{y}}(U)} \frac{f_j(x_j) + f_k(x_k) - f_i(x_i)}{f_i(x_i) f_j(x_j) f_k(x_k)},$$

so

$$C_{\mathbf{y}} = \lim_{U\to\infty} C_U.$$

Then

$$\begin{aligned} C_U &= C_{\mathbf{y}} - \frac{3}{\pi^2} \sum_{\substack{\mathbf{x}\in\mathfrak{T}_{\mathbf{y}} \\ H(\mathbf{x})\geq U}} \frac{f_j(x_j) + f_k(x_k) - f_i(x_i)}{f_i(x_i) f_j(x_j) f_k(x_k)} \\ &= C_{\mathbf{y}} + O\left(\sum_{\substack{\mathbf{x}\in\mathfrak{T}_{\mathbf{y}} \\ H(\mathbf{x})\geq U}} \frac{1}{x_i^2 f_i(x_i) f_j(x_j) f_k(x_k)} \right) \end{aligned}$$

where in the last line we have used the result in Eq. 9. To evaluate the tail of this sum, we introduce the quantity

$$N'_{\mathbf{y}}(T) = \#\{\mathbf{x}\in\mathfrak{T}_{\mathbf{y}} : x_i < T\}.$$

Since $x_i < H(\mathbf{x}) < 6a_3 x_i$, we know $N'_{\mathbf{y}}(T) = O(\log^2 T)$ by Corollary 2.3. Note also that $a_i x_i < d x_j x_k \leq d x_j^2$, so both $f_i(x_i)$ and $f_j(x_j)$ are bounded below by a constant times $\log x_i$. Note that $f_k(x_k) \geq f_k(w_k) > 0$, so it is bounded below.

Thus,

$$O\left(\sum_{\substack{\mathbf{x}\in T_{\mathbf{y}}\\ H(\mathbf{x})\geq U}}\frac{1}{x_i^2 f_i(x_i)f_j(x_j)f_k(x_k)}\right)=O\left(\sum_{\substack{\mathbf{x}\in T_{\mathbf{y}}\\ H(\mathbf{x})\geq U}}\frac{1}{x_i^2\log^2 x_i}\right)$$
$$=O\left(\sum_{\substack{\mathbf{x}\in T_{\mathbf{y}}\\ x_i\geq U/6a_3}}\frac{1}{x_i^2\log^2 x_i}\right).$$

For simplicity, let us set $U'=\lfloor U/6a_3\rfloor$, where $\lfloor x\rfloor$ is the greatest integer function. Then, we have

$$O\left(\sum_{\substack{\mathbf{x}\in T_{\mathbf{y}}\\ H(\mathbf{x})\geq U}}\frac{1}{x_i^2 f_i(x_i)f_j(x_j)f_k(x_k)}\right)=O\left(\sum_{n=U'}^{\infty}\frac{N'_{\mathbf{y}}(n+1)-N'_{\mathbf{y}}(n)}{n^2\log^2 n}\right)$$
$$=O\left(-\frac{N'_{\mathbf{y}}(U'-1)}{(U')^2\log^2 U'}+\sum_{n=U'}^{\infty}N'_{\mathbf{y}}(n+1)\left(\frac{1}{(n-1)^2\log^2(n-1)}-\frac{1}{n^2\log^2 n}\right)\right)$$
$$=O\left(\frac{N'_{\mathbf{y}}(U)}{U^2\log^2 U}\right)+O\left(\sum_{n=U'}^{\infty}\frac{N'_{\mathbf{y}}(n+1)}{n^3\log^2 n}\right)$$
$$=O\left(\frac{1}{U^2}\right).$$

This establishes that $C_{\mathbf{y}}$ exists and that $C_U=C_{\mathbf{y}}+O(1/U^2)$. This also allows us to estimate D_U, since

$$D_U=O\left(\frac{1}{U^2}C_U\right)=O\left(\frac{1}{U^2}\right).$$

To estimate E_U, we note that, by Eq. 15,

$$E_U=O\left(\frac{1}{f_k(y_k)}+\sum_{\mathbf{x}\in\mathfrak{T}_{\mathbf{y}}(U)}\frac{1}{f_j(x_j)}\right).$$

We again note that $O(1/f_j(x_j))=O(1/\log(x_i))$, so

$$E_U=O(1)+O\left(\sum_{n=1}^{\lfloor U\rfloor}\frac{N'_{\mathbf{y}}(n)-N'_{\mathbf{y}}(n-1)}{\log n}\right)$$
$$=O(1)+O\left(\frac{N'_{\mathbf{y}}(U)}{\log\lfloor U\rfloor}+\sum_{n=1}^{\lfloor U\rfloor-1}N'_{\mathbf{y}}(n)\left(\frac{1}{\log n}-\frac{1}{\log(n+1)}\right)\right)$$
$$=O(1)+O\left(\frac{\log^2 U}{\log U}\right)+O\left(\sum_{n=1}^{\lfloor U\rfloor-1}\frac{N'_{\mathbf{y}}(n)}{n\log^2 n}\right)$$
$$=O(\log U).$$

Combining these results in Eq. 13, we get

$$N_{\mathbf{y}}(T) = C_{\mathbf{y}} \log^2 T + O\left(\log^2 U + \frac{\log^2 T}{U^2} + \log U \log T \log\log T\right).$$

To make the error as small as possible, we choose $U = \frac{\sqrt{\log T}}{\log\log T}$, which gives

$$N_{\mathbf{y}}(T) = C_{\mathbf{y}} \log^2 T + O\left(\log T (\log\log T)^2\right). \qquad \square$$

3. The Rosenberger variations again

In this section, we establish the conditions of Theorems 2.2 and 2.4 for each of the Rosenberger equations R_1, R_2, and R_3.

Given a solution $\mathbf{x}$ to a Rosenberger equation, we often want to select the component x_i of $\mathbf{x}$ such that $H(\phi_i(\mathbf{x})) \leq H(\mathbf{x})$. This component is almost always the largest component:

Lemma 3.1. *Suppose* $\mathbf{x}$ *is a positive integer solution to the Rosenberger equation* R_1, R_2, *or* R_3. *Let* (i, j, k) *be the permutation of* $\{1, 2, 3\}$ *such that* $H(\phi_i(\mathbf{x})) \leq H(\mathbf{x})$ *and* $x_j \geq x_k$. *Then either*

$$x_i \geq x_j$$

or $x_k = 1$ *in* R_3, *and* $k = 1$ *or* 2.

Proof. Let

$$f(T) = a_i T^2 + a_j x_j^2 + a_k x_k^2 - dT x_j x_k.$$

Then $f(x_i) = 0$. Let x_i' be the other root of $f(T)$. Since $H(\phi_i(\mathbf{x})) \leq H(\mathbf{x})$, we know $x_i' \leq x_i$. We consider

$$f(x_j) = a_i x_j^2 + a_j x_j^2 + a_k x_k^2 - d x_j^2 x_k \leq (a_1 + a_2 + a_3 - d x_k) x_j^2. \tag{16}$$

If $x_k \geq 2$, then the right hand side of Eq. 16 is negative, so $f(x_j) < 0$ and hence, $x_i' < x_j < x_i$. If $x_k = 1$, then the right hand side is zero for equations R_1 and R_2, so in these cases, $x_j \leq x_i$. If $x_3 = 1$ in R_3, then $k = 3$ and

$$x_i' = 5x_j - x_i \leq x_i,$$

which implies $x_i > x_j$. $\square$

Note that, if x_1 or $x_2 = 1$ in R_3, and $H(\phi_3(\mathbf{x})) < H(\mathbf{x})$, then x_3 is in fact not the maximal component of $\mathbf{x}$. This creates a minor wrinkle in the study of R_3.

It will also be useful to note that, if $x_i \geq x_j$, x_k, then

$$\begin{aligned} a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 &= d x_1 x_2 x_3 \\ a_1 x_i^2 + a_2 x_i^2 + a_3 x_i^2 &\geq d x_i x_j x_k \\ x_i &\geq \frac{d}{a_1 + a_2 + a_3} x_j x_k. \end{aligned}$$

For equations R_1 and R_2, this gives

$$x_i \geq x_j x_k \tag{17}$$

and for R_3, this yields

$$x_i \geq \frac{5}{7} x_j x_k. \tag{18}$$

Let us now consider the various cases. For these cases, it will be more convenient to set $(x, y, z) = (x_1, x_2, x_3)$.

Equation R_1: Let us first suppose that $i = 1$. Then $x \geq yz$ by Eq. 17. We can improve on this bound by noting that $y,\, z \geq 1$, and observing that

$$\begin{aligned}(x-yz)(x-3yz) &= x^2 - 4xyz + 3y^2z^2\\ &= x^2 - x^2 - y^2 - 2z^2 + 3y^2z^2\\ &= y^2(z^2-1) + 2z^2(y^2-1) \geq 0.\end{aligned}$$

If we have equality, then $y = z = 1$ and $x = 1$ or 3. If $x = 1$, then $i \neq 1$, and if $x = 3$, then $x = 3yz$. Otherwise, $x - yz > 0$ so $x - 3yz > 0$. That is, $x \geq 3yz$ for all $\mathbf{x}$ with $i = 1$. Hence, we may choose $r_1 = 4/3$ (in Theorem 2.4). By symmetry, we may also choose $r_2 = 4/3$.

If $i = 3$, then we may assume, without loss of generality, that $y \geq x$. By observing how the tree of solutions begins (see Figure 2) and excluding $(1, 1, 1)$, we may further assume that $y \geq 3$ and $x \geq 1$. Now observe that

$$2\left(z - \frac{1}{3}xy\right)\left(z - \frac{5}{3}xy\right) = \frac{1}{9}x^2(y^2-9) + y^2(x^2-1) \geq 0.$$

Thus, since $z \geq xy > \frac{1}{3}xy$, we get $z \geq \frac{5}{3}xy$ for all $\mathbf{x} \neq (1, 1, 1)$ with $i = 3$. This yields $r_3 = \frac{12}{5}$.

Equation R_2: If $i = 1$, then $x \geq y \geq 1$ and $x \geq z \geq 1$, so

$$(x-yz)(x-5yz) = 2y^2(z^2-1) + 3z^2(y^2-1) \geq 0.$$

We have equality if and only if $\vec{x} = (1, 1, 1)$ or $(5, 1, 1)$. If $x = 1$ then $i \neq 1$, and for $\mathbf{x} = (5, 1, 1)$, $x = 5yz$. Otherwise, $x > 5yz$, so we may choose $r_1 = \frac{6}{5}$.

If $i = 2$, then $y \geq x \geq 1$ and $y \geq z \geq 1$, so

$$2(y-xz)(y-2xz) = x^2(y^2-1) + 3z^2(x^2-1).$$

We have equality only if $(x, y, z) = (1, 1, 1)$ or $(1, 2, 1)$. If $y = 1$ then $i \neq 2$, and if $y = 2$ then $y = 2xz$. Otherwise, $y \geq 2xz$, so we may choose $r_2 = 3$.

We will split the case when $i = 3$ up into two cases. First, let us suppose $z \geq y \geq x$ and $\mathbf{x} \neq (1, 1, 1)$. Then we may assume $y \geq 2$ and $x \geq 1$ (see Figure 3), so

$$3\left(z - \frac{1}{2}xy\right)\left(z - \frac{3}{2}xy\right) = \frac{1}{4}x^2(y^2-4) + 2y^2(x^2-1) \geq 0.$$

Thus, $z \geq \frac{3}{2}xy$. If $z \geq x \geq y$, then we may assume $x \geq 5$ and $y \geq 1$ (see Figure 3), so

$$3\left(z - \frac{1}{5}xy\right)\left(z - \frac{9}{5}xy\right) = x^2(y^2-1) + \frac{2}{25}y^2(x^2-25) \geq 0.$$

Thus, $z \geq \frac{9}{5}xy$. Combining these two inequalities, we get $z \geq \frac{3}{2}xy$ whenever $i = 3$ and $\mathbf{x} \neq (1, 1, 1)$. Thus, we may choose $r_3 = 4$.

Equation R_3: If $i = 1$ and $y \geq 2$, then $x \geq \frac{5}{7}yz$ by Lemma 3.1 and Eq. 18. Note also that

$$\left(x - \frac{1}{2}yz\right)\left(x - \frac{9}{2}yz\right) = y^2(z^2-1) + \frac{5}{4}z^2(y^2-4) \geq 0,$$

so $x > \frac{9}{2}yz$. The solutions where $y = 1$ are all on the branch rooted at $(2, 1, 1)$ and generated by ϕ_1 and ϕ_3:

$$(2,1,1) \longrightarrow (3,1,1) \longrightarrow (3,1,2) \longrightarrow (7,1,2) \longrightarrow (7,1,5) \longrightarrow \ldots. \tag{19}$$

Every other solution on this branch, starting with $(3,1,1)$, has $i = 1$. So let us write

$$(x_n, 1, z_n) = (\phi_1\phi_3)^n(3,1,1).$$

We claim that $x_n \geq 3z_n$, and prove this using induction. It is clearly true for the case when $n = 0$. Note that

$$(x_{n+1}, 1, z_{n+1}) = \phi_1\phi_3(x_n, 1, z_n) = (4x_n - 5z_n, 1, x_n - z_n),$$

and

$$4x_n - 5z_n \geq 3(x_n - z_n)$$

if and only if $x_n \geq 2z_n$. The latter is true by our induction hypothesis, so $x_n \geq 3z_n$ for all n. Combining this with the case when $y \geq 2$, we get $x \geq 3yz$ whenever $i = 1$, and hence we can choose $r_1 = \frac{5}{3}$. By symmetry, we may choose $r_2 = \frac{5}{3}$, too.

If $i = 3$ and $y \geq 2$, then $z \geq \frac{5}{7}xy$ by Lemma 3.1 and Eq. 18. If $y = 1$, then $\mathbf{x}$ is in the branch described above in Eq. 19. In a similar fashion, let us write

$$(x_n, 1, z_n) = (\phi_3\phi_1)^n(2,1,1).$$

We claim that $z_n \geq \frac{2}{3}x_n$ for $n \geq 1$. We note that $(x_1, 1, z_1) = (3,1,2)$, so the claim is true in the base case. We note that

$$(x_{n+1}, 1, z_{n+1}) = \phi_3\phi_1(x_n, 1, z_n) = (5z_n - x_n, 1, 4z_n - x_n)$$

and

$$\begin{aligned} 4z_n - x_n &\geq \frac{2}{3}(5z_n - x_n) \\ 12z_n - 3x_n &\geq 10z_n - 2x_n \\ 2z_n &\geq x_n. \end{aligned}$$

The last is true since $2z_n \geq \frac{4}{3}x_n$, by the induction hypothesis. Thus, if $i = 3$, then $z \geq \frac{2}{3}xy$ for $\mathbf{x} \neq (2,1,1)$ or $(1,2,1)$. Hence, we may choose $r_3 = \frac{15}{2}$.

4. Calculations

We calculate C_m using the formula for C_U in Eq. 14. For Equation R_1, using $U = 10^6$ we find

$$C_{(1,1,1)} \approx 0.543809447296.$$

Calculations using $U = 10^{10}$ appear to be accurate to 22 digits and take about a second of computing time (using a 500Mhz Celeron). The constant C_1 in Theorem 0.1 is obtained by multiplying by 3, to account for the solutions with negative entries. For Equation R_2, using $U = 10^6$, we find

$$C_{(1,1,1)} \approx .554239131152.$$

The constant C_2 is 3 times this. Finally, for equation R_3, we use $U = 10^7$ and find

$$C_{(1,2,1)} \approx .588051990717.$$

The constant C_3 is 6 times this, to account for solutions with negative entries and solutions in the tree $\mathfrak{T}_{(2,1,1)}$.

5. Applications to other equations

There are several places within the above discussions where we have made use of certain properties of the Rosenberger variations. Specifically, we made use of the following:

(1) If $\mathbf{x}$ is a positive integral solution, than so is $\phi_i(\mathbf{x})$ for $i = 1$, 2, and 3.
(2) Descent, when it occurs, is unique.
(3) If $\mathbf{x}$ is an integer solution and $x_i = 0$ for some i, then $\mathfrak{T}_{\mathbf{x}}$ is finite.

For a particular equation of the form

$$ax^2 + by^2 + cz^2 = dxyz + e, \tag{20}$$

these properties are no doubt easy to verify, but general results seem overly complicated and not worth pursuing. If the integer solutions in a tree of solutions for an equation of the form Eq. 20 satisfy these properties, then we may apply Theorem 2.4 to that tree, though some of our arguments may have to be modified (for example, if e is large enough, then the inequalities in Lemma 2.1 change directions). These properties are easy enough to check for the equations studied in [J-S], where $e = 1$. Thus, one need only check the conditions of Theorem 2.4 and calculate C_U for large enough U. We have done this, but spare the reader the details. The conditions of Theorem 2.4 are the most difficult items to check. As a consequence, we have the following theorem:

Theorem 5.1. *Let*

$$N(T) = \#\{\mathbf{x} = (x, y, z) \in \mathbb{Z}^3 : ax^2 + by^2 + cz^2 = dxyz + 1 \text{ and } H(\mathbf{x}) < T\}.$$

Then, for the equations listed in Table 1,

$$N(T) = C \log^2 T + O(\log T(\log\log T)^2)$$

where approximations for C are also given.

Equation	Fundamental solution(s)	C
$x^2 + 5y^2 + 5z^2 = 5xyz + 1$	$(4, 1, 2)$ and $(4, 2, 1)$	3.92062681166
$x^2 + 3y^2 + 6z^2 = 6xyz + 1$	$(2, 1, 1)$	2.22381295435
$2x^2 + 7y^2 + 14z^2 = 14xyz + 1$	$(2, 1, 1)$	1.85092947320
$2x^2 + 2y^2 + 3z^2 = 6xyz + 1$	$(1, 1, 1)$	3.04230700308
$6x^2 + 10y^2 + 15z^2 = 30xyz + 1$	$(1, 1, 1)$	1.86988733010
$x^2 + 2y^2 + 2z^2 = 2xyz + 1$	$(3, 2, 2)$	3.69061353513

Table 1. Equations of the form of Eq. 5, together with the constants C, accurate to 12 places. The constant C was calculated using $U = 10^6$.

For equations of the form

$$x^2 + by^2 + bz^2 = 2bxyz + 1, \tag{21}$$

the conditions of Theorem 2.4 are probably satisfied for every fundamental solution $(1, y, y)$, but the number of fundamental solutions with height less than T grows asymptotically like $O(T)$. In [B2], such rapid growth was also noted for $b = 1$ in Eq. 21, and for the equations

$$x^2 + y^2 + z^2 = dxyz + e,$$

where $(d,e) = (1, s^2+4)$ or $(2, s^2+1)$ and $s \in \mathbb{Z}$.

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