The Calculus of Parametric Curves

1. dy/dx and also d^2y/dx^2 for each of the following parametrically described curves at the given value of the parameter. You may wish to check your results by eliminating the parameter, if possible, and re-working the problem as a non-parametric problem. You should also use your graphing calculator to examine the graphs and judge whether your calculated information looks graphically plausible.

   (a) $x = t + 1, y = t^2 + 3t \; : t = 3$

   (b) $x = \sqrt{t}, \; y = \sqrt{t - 1} \; : t = 2$

   (c) $x = \cos^3 \theta, \; y = \sin^3 \theta$

2. Find the slope of the tangent line to the curves below at the given points:

   (a) $x = 2\cot\theta, \; y = 2\sin^2\theta$ ***at*** $(0,2)$ and $(2\sqrt{3}, 1/2)$

   (b) $x = 2 - 3\cos\theta, \; y = 3 + 2\sin\theta$ ***at*** $(2,5)$

3. Find all points of horizontal and vertical tangency to the curves below:

   (a) $x = 1 - t, \; y = t^2$

   (b) $x = 1 - t, \; y = t^3 - 3t$

   (c) $x = \cos t + t\sin t, \; y = \sin t - t\cos t$ on $[0, 3\pi]$

4. Find the arc-length of the following parametric curves over the given intervals:

(a) $x = e^{-t}\cos t,\ y = e^{-t}\sin t$: $[0, 2\pi]$

(b) $x = t^2,\ y = 2t$: $[0, 2]$

(c) $x = t,\ y = \dfrac{t^5}{10} + \dfrac{1}{6t^3}$: $[1, 2]$

5. Find the surface area generated by revolving the given curve about the given axis:

(a) $x = t,\ y = 2t$, revolved about x - axis, $0 \le t \le 4$

(b) same curve as in (a) revloved about y - axis

(c) $x = 4\cos t,\ y = 4\sin t,\ 0 \le t \le \pi/2$, revolved about y - axis

6. Find the area of the region in the first quadrant bounded by the given parametric curve , and the line x=2:

$$x = 2\sin^2 t,\ y = 2\sin^2 t \tan t,\ 0 \le t \le \pi/2$$

7. Find the area of the region bounded by the given parametric curve and the entire x-axis:

$$x = 2\cot t,\ y = 2\sin^2 t,\ 0 < t < \pi$$

Solutions for Calculus of Parametric Curves

(1) **(a)** $x = t+1,\ y = t^2 + 3t\ ;\ t = 3.$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+3}{1} = 2t+3$$, so when $t=3$,

$$\frac{dy}{dx} = 2(3)+3 = 9.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(2t+3) = \frac{\frac{d}{dt}(2t+3)}{dx/dt}$$

$$= \frac{2}{1} = 2.$$

(b) $x = \sqrt{t},\ y = \sqrt{t-1}\ ;\ t = 2$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{2\sqrt{t-1}}}{1/2\sqrt{t}} = \frac{\sqrt{t}}{\sqrt{t-1}}$$

when $t=2$, $dy/dx = \frac{\sqrt{2}}{\sqrt{2-1}} = \sqrt{2}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\sqrt{t}}{\sqrt{t-1}}\right) = \frac{\frac{d}{dt}\left(\sqrt{t}/\sqrt{t-1}\right)}{1/2\sqrt{t}}$$

$$= \frac{\sqrt{t-1}\cdot\frac{1}{2\sqrt{t}} - \sqrt{t}\cdot\frac{1}{2\sqrt{t-1}}}{t-1} \cdot \frac{2\sqrt{t}}{1}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=2} = -1$$

(c) $x = \cos^3\theta$, $y = \sin^3\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}, \quad \frac{dy}{d\theta} = 3\sin^2\theta\cos\theta,$$

$$\frac{dx}{d\theta} = -3\cos^2\theta\sin\theta \quad \text{so } \frac{dy}{dx} = \frac{3\sin^2\theta\cos\theta}{-3\sin\theta\cos^2\theta}$$

$$= -\tan\theta$$

$$\therefore \left.\frac{dy}{dx}\right|_{\theta=\pi/4} = -\tan\pi/4 = -1$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-\tan\theta) = \frac{\frac{d}{d\theta}(-\tan\theta)}{dx/d\theta} = \frac{-\sec^2\theta}{-3\cos^2\theta\sin\theta}$$

$$= \frac{1}{3\cos^4\theta\sin\theta} \qquad \therefore \left.\frac{d^2y}{dx^2}\right|_{\theta=\pi/4} = \frac{1}{3(1/\sqrt{2})\cdot(1/\sqrt{2})}$$

$$= \frac{4\sqrt{2}}{3}$$

2 **(a)** note $(0,2)$ occurs for $\theta = \pi/2$ and $(2\sqrt{3}, 1/2)$ occurs when $\theta = \pi/6$. $\frac{dy}{dx} = \frac{4\sin\theta\cos\theta}{-2\csc^2\theta} =$

$$-2\cos\theta\sin^3\theta \quad \therefore \left.\frac{dy}{dx}\right|_{\substack{(0,2)\\ \theta=\pi/2}} = 0 \text{ and } \left.\frac{dy}{dx}\right|_{\theta=\pi/6} = -\sqrt{3}/8$$

13

(b) $x = 2 - 3\cos\theta$, $y = 3 + 2\sin\theta$ at $(2,5)$.

Note $(2,5)$ occurs when $\theta = \pi/2$.

$dy/d\theta = 2\cos\theta$, $dx/d\theta = 3\sin\theta$

$\therefore dy/dx = \dfrac{2\cos\theta}{3\sin\theta} = \frac{2}{3}\cot\theta$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \frac{2(0)}{3(1)} = 0$$

Now plot the curve & see that this appears correct.

3 (a) $x = 1-t$, $y = t^2$. $\dfrac{dx}{dt} = -1$, $\dfrac{dy}{dt} = 2t$

$\dfrac{dy}{dt} = 0$ when $t = 0$ (then also $\dfrac{dx}{dt} \neq 0$)

so $\left.\begin{array}{l} x = 1 - 0 = 1 \\ y = 0^2 = 0 \end{array}\right\} \longrightarrow (1,0)$ is only point of horizontal tangency. Since $\dfrac{dx}{dt}$ is never zero, no points of vertical tangency.

(b) $x = 1-t$, $y = t^3 - 3t$. $\dfrac{dx}{dt} = -1$, $\dfrac{dy}{dt} = 3t^2 - 3$

so $\dfrac{dy}{dt} = 0$ when $t = \pm 1$, and $dx/dt \neq 0$.

so $(x(1), y(1)) = (0,-2)$ and $(x(-1), y(-1)) = (2,2)$ are points of horizontal tangency. No points of vertical tangency

(c) $x = \cos t + t\sin t$, $y = \sin t - t\cos t$ on $[0, 3\pi]$

$\frac{dx}{dt} = t\cos t$, $\frac{dy}{dt} = t\sin t$

$dy/dt = 0$ when $t = 0, \pi, 2\pi, 3\pi$

note $\frac{dy}{dt} = \frac{dx}{dt} = 0$ only when $t = 0$.

$$\lim_{t\to 0} \frac{dy}{dx} = \lim_{t\to 0} \frac{dy/dt}{dx/dt} = \lim_{t\to 0} \frac{t\sin t}{t\cos t} = \lim_{t\to 0} \tan t = 0$$

so the tangent lines "flatten out" as one moves into the initial point.

Also $\frac{dx}{dt} = 0$ for $t = \pi/2, 3\pi/2, 5\pi/2$

so points of horizontal tangency occur for $t = 0, \pi, 2\pi, 3\pi$

and points of vertical tangency occur for $t = \pi/2, 3\pi/2, \frac{5\pi}{2}$.

(4) (a) $x = e^{-t}\cos t$, $y = e^{-t}\sin t$; $[0, \pi/2]$.

$$\text{Arc Length} = \int_{t=0}^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + (dy/dt)^2}\, dt =$$

$$\int_0^{\pi/2} \sqrt{[e^{-t}(-\sin t) - e^{-t}\cos t]^2 + [e^{-t}\cos t - e^{-t}\sin t]^2}\, dt$$

$$= \int^{\pi/2} e^{-t}\sqrt{2} \quad = -\sqrt{2}(e^{-\pi/2} - 1)$$

15

(b) $x = t^2, \; y = 2t \; ; \; [0,2]$

$\frac{dx}{dt} = 2t, \; dy/dt = 2$

$\therefore$ Arc length $= \int_0^2 \sqrt{4t^2 + 4}\, dt = 2\int_0^2 \sqrt{t^2+1}\, dt$

$\doteq$ 5.916 (by Electronic integration utility), or one can integrate explicitly via the trig subst. $t = \tan\theta$, etc.

(c) $\frac{dx}{dt} = 1, \; \frac{dy}{dt} = \frac{t^4}{2} - \frac{t^{-4}}{2}$. $\therefore \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 =$

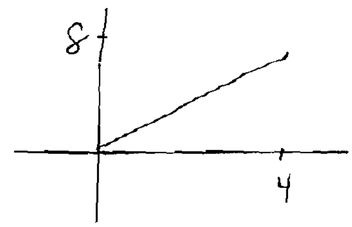
$1 + \left[\frac{t^4 - t^{-4}}{2}\right]^2 = 1 + \frac{t^8 - 2 + t^{-8}}{4} = \frac{t^8 + 2 + t^{-8}}{4}$

So Arc length $= \int_1^2 \sqrt{(t^8 + 2 + t^{-8})/4}\, dt =$

$\frac{1}{2}\int_1^2 \sqrt{(t^4 + t^{-4})^2}\, dt = \frac{1}{2}\int_1^2 (t^4 + t^{-4})\, dt$

$= \frac{1}{2}\left[\frac{t^5}{5} + \frac{t^{-3}}{-3}\right]_1^2 \doteq$ 3.246

5 (a) $x = t,\ y = 2t \quad 0 \le t \le 4$

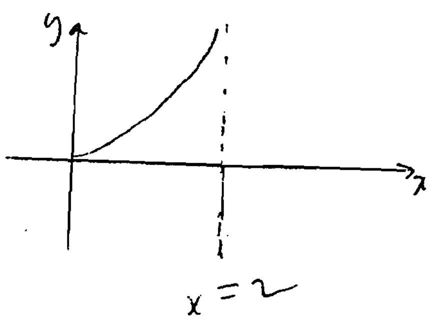


$$S.A. = \int_0^4 2\pi(2t)\sqrt{1^2+2^2}\,dt = 32\pi\sqrt{5}$$

(b) $$S.A. = \int_0^4 2\pi t\sqrt{5}\,dt = 16\pi\sqrt{5}$$

(c) $$S.A. = \int_0^{\pi/2} 2\pi(4\cos t)\sqrt{(-4\sin t)^2+(4\cos t)^2}\,dt$$
$$= 32\pi.$$

6



$$x = 2\sin^2 t \qquad 0 \le t < \pi/2$$
$$y = 2\sin^2 t\tan t$$

$$dx = 4\sin t\cos t\,dt$$

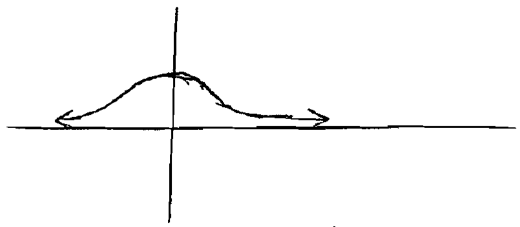
$$\text{Area} = \int_{x=0}^{2} y\,dx = \int_{t=0}^{\pi/2} (2\sin^2 t\tan t)(4\sin t\cos t)\,dt$$

$$= 8\int_0^{\pi/2}\sin^4 t\,dt = 3\pi/2 \approx 4.712$$

[to integrate, use $\sin^2\theta = \frac{1-\cos 2\theta}{2}$ several times].

(7) $x = 2\cot t \;,\; y = 2\sin^2 t \;,\; 0 < t < \pi$

$$dx = -2\csc^2 t$$



$$\text{Area} = 2\int_{x=0}^{\infty} y\,dx = 2\int_{\pi/2}^{0} 2\sin^2 t\,(-2\csc^2 t)\,dt$$

$$= 8\int_{0}^{\pi/2} dt = 4\pi$$