Chaining it all together...

1. Find dw/dt by using the appropriate Chain Rule:

   $(a)\ w = x^2 + y^2, x = e^t,\ y = e^{-t}$

   $(b)\ w = x \sec y, x = e^t, y = \pi - t$

   $(c)\ w = xy, x = 2\sin t,\ y = \cos t$

   $(d)\ w = x^2 + y^2 + z^2, x = e^t \cos t,\ y = e^t \sin t,\ z = e^t$

   $(e)\ w = xy + xz + yz,\ x = t - 1,\ y = t^2 - 1,\ z = t$

2. Find the first partial derivatives with respect to s and t, and evaluate at the specified point, by using the appropriate chain rule:

   $(a)\ w = x^2 + y^2, x = s + t, y = s - t, s = 2, t = -1$

   $(b)\ w = x^2 - y^2, x = s\cos t,\ y = s\sin t,\ s = 3,\ t = \pi / 4$

3. Find the partial derivatives with respect to r and θ:

   $(a)\ w = x^2 - 2xy + y^2, x = r + \theta,\ y = r - \theta$

   $(b)\ w = \arctan(y / x),\ x = r\cos\theta,\ y = r\sin\theta$

4. Differentiate implicitly to find dy/dx (Calculus I review):

   $(a)\ \sin x + \sec xy - 3 = 0$

   $(b)\ \ln\sqrt{x^2 + y^2} + xy = 4$

5. Differentiate implicitly to find the first partials of z:

   $(a)\ x^2 + y^2 + z^2 = 25$

   $(b)\ \tan(x + y) + \tan(y + z) = 1$

   $(c)\ e^{xz} + xy = 0$

Solutions for Chaining it all together

(a)

1) $w = x^2+y^2, \ x = e^{t}, \ y = e^{-t}$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = 2x\cdot e^{t} + 2y(-e^{-t}) =$$

$$2e^{t}\cdot e^{t} + -2e^{-t}\cdot e^{-t} = 2e^{2t} - 2e^{-2t}$$

(b) $W = x\sec y, \ x = e^{t}, \ y = \pi - t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = (\sec y)\cdot e^{t} + x\sec y\tan y\,(-1)$$

$$= e^{t}\sec(\pi-t) + -e^{t}\sec(\pi-t)\tan(\pi-t)$$ or using

and the trig identity $\sec(\pi-t) = -\sec t$ one can write

$$\frac{dw}{dt} = -e^{t}\sec t - e^{t}\sec t\tan t$$

(c) $w = xy, \ x = 2\sin t, \ y = \cos t$

$$\frac{dw}{dt} = w_x\frac{dx}{dt} + w_y\frac{dy}{dt} = y\cdot(2\cos t) + x(-\sin t)$$

$$= 2\cos^2 t - 2\sin^2 t$$ or using a trig double-angle

identity, $\frac{dw}{dt} = 2[\cos^2 t - \sin^2 t] = 2\cos 2t$

(d) $w = x^2+y^2, \ x = e^{t}\cos t, \ y = e^{t}\sin t, \ z = e^{t}$

$$\frac{dw}{dt} = w_x\frac{dx}{dt} + w_y\frac{dy}{dt} + w_z\frac{dz}{dt} =$$

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$$2x\,x' + 2y\,y' + 2z\,z' =$$

$$2(e^t\cos t)\left[-e^t\sin t + e^t\cos t\right] + 2e^t\sin t\left[e^t\cos t + e^t\sin t\right] + 2e^t\cdot e^t$$

$$= 2e^{2t}\cos t[\cos t - \sin t] + 2e^{2t}\sin t[\cos t + \sin t] + 2e^{2t}$$

$$= 2e^{2t}\cos^2 t\ 2e^{2t}\sin^2 t + 2e^{2t}$$

$$= 2e^{2t}\left[\underbrace{\cos^2 t + \sin^2 t}_{1} + 1\right] = 4e^{2t}$$

(e) $w = xy + xz + yz$, $x = t-1$, $y = t^2-1$, $z = t$

$$\frac{dw}{dt} = w_x x' + w_y y' + w_z z' = (y+z) + (x+z)(2t) + (x+y)$$

$$= \left[(t^2-1)+t\right] + \left[(t-1)+t\right](2t) + \left[(t-1)+(t^2-1)\right]$$

$$= 6t^2 - 3$$

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2 (a) $w = x^2+y^2$, $x = s+t$, $y = s-t$

$$\frac{\partial w}{\partial s} = w_x\frac{\partial x}{\partial s} + w_y\frac{\partial y}{\partial s} = 2x\cdot 1 + 2y\cdot 1 =$$

$$2(s+t) + 2(s-t) = 2(2+-1) + 2(2-(-1)) = 8$$

$$\frac{\partial w}{\partial t} = w_x\frac{\partial x}{\partial t} + w_y\frac{\partial y}{\partial t} = 2x - 2y = 2(s+t) - 2(s-t) =$$

$$2(2+-1) - 2(2-(-1)) = -4$$

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2(b) $w = x^2 - y^2$, $x = s\cos t$, $y = s\sin t$, $s = 3$, $t = \frac{\pi}{4}$

$$\frac{\partial w}{\partial s} = w_x \frac{\partial x}{\partial s} + w_y \frac{\partial y}{\partial s} = 2x\cos t + -2y\sin t$$

$$= 2s\cos^2 t - 2s\sin^2 t =$$

$$2(3)\cos^2\frac{\pi}{4} - 2(3)\sin^2\frac{\pi}{4} =$$

$$6\left[\left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2\right] = 6\cdot 0 = 0$$

$$\frac{\partial w}{\partial t} = w_x \frac{\partial x}{\partial t} + w_y \frac{\partial y}{\partial t} = 2x(-s\sin t) + -2y(s\cos t)$$

$$= -2s^2\sin t\cos t + -2s^2\sin t\cos t = -4s^2\sin t\cos t$$

$$= -4(3)^2\left(\frac{1}{\sqrt{2}}\right)^2 = -18$$

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3 (a) $w = x^2 - 2xy + y^2$, $x = r + \theta$, $y = r - \theta$

$$\frac{\partial w}{\partial r} = w_x \frac{\partial x}{\partial r} + w_y \frac{\partial y}{\partial r} =$$

$$(2x - 2y) + (-2x + 2y) = 0$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} = (2x - 2y) + (-2x + 2y)(-1)$$

$$= 4x - 4y = 4(r + \theta) - 4(r - \theta)$$

$$= 8\theta$$

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(b) $w = \arctan(y/x)$; $x = r\cos\theta,\ y = r\sin\theta$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} =$$

$$-\frac{1}{x^2}\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot y\cdot(-r\sin\theta) + \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}\cdot r\cos\theta$$

$$= \frac{ry\sin\theta}{x^2+y^2} + \frac{r\cos\theta}{x(1+y^2/x^2)}$$

$$= \frac{r\cdot r\sin\theta\cdot\sin\theta}{x^2+y^2} + \frac{\cancel{r\cos\theta}}{\cancel{r\cos\theta}(1+y^2/x^2)}$$

$$= \frac{r^2\sin^2\theta}{x^2+y^2} + \frac{1}{1+\frac{y^2}{x^2}} =$$

$$\frac{r^2\sin^2\theta}{x^2+y^2} + \frac{x^2}{x^2+y^2} = \frac{r^2\sin^2\theta + r^2\cos^2\theta}{x^2+y^2}$$

$$= \frac{r^2}{r^2} = 1$$

$$\frac{\partial w}{\partial r} = w_x\frac{\partial x}{\partial r} + w_y\frac{\partial y}{\partial r} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{-y}{x^2}\cos\theta + \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}\sin\theta$$

$$= \frac{-y\cos\theta}{x^2+y^2} + \frac{x\sin\theta}{x^2+y^2} = \frac{-r\sin\theta\cos\theta + r\sin\theta\cos\theta}{r^2}$$

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(a) $\sin x + \sec xy - 3 = 0$

~~Applying~~ the operator $\frac{d}{dx}$ and regarding y as a function of x (locally) : $\cos x + \sec xy \tan xy\,[xy' + y] = 0$

so
$$\frac{\dfrac{-\cos x}{\sec xy \tan xy} - y}{x} = y'$$

$$y' = \frac{-\cos x \cos xy \cot xy - y}{x} \quad \text{etc.}$$

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$$\ln(x^2+y^2)^{1/2} + xy = 4$$
$$\frac{1}{2}\ln(x^2+y^2) + xy = 4$$
$$\frac{1}{\cancel{2}} \cdot \frac{1}{x^2+y^2}\,(\cancel{2}x + \cancel{2}y\,y') + xy' + y = 0$$

mult by x^2+y^2 gives: $x + yy' + (x^2+y^2)xy' + y(x^2+y^2) = 0$

$$yy' + x^3y' + xy^2y' = -y(x^2+y^2) - x$$
$$y'(y + x^3 + xy^2) = -y(x^2+y^2) - x$$
$$y' = \frac{-y(x^2+y^2) - x}{y + x^3 + xy^2}$$

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(a) $x^2+y^2+z^2=25$

To find $\frac{\partial z}{\partial x}$, apply $\frac{\partial}{\partial x}$ operator, where z is the depend

variable, and x and y are both independent variables, so

both $\frac{\partial x}{\partial y}$ and $\frac{\partial y}{\partial x}$ are zero:

$$2x + 2z\frac{\partial z}{\partial x} = 0 \Longrightarrow \frac{\partial z}{\partial x} = -\frac{x}{z}$$

To find $\frac{\partial z}{\partial y}$: $2y + 2z\frac{\partial z}{\partial y} = 0 \Longrightarrow \frac{\partial z}{\partial y} = -y/z$

(b) $\tan(x+y) + \tan(y+z) = 1$. Applying $\frac{\partial}{\partial x}$:

$$\sec^2(x+y) + \sec^2(y+z)\cdot\frac{\partial z}{\partial x} = 0$$

$$\therefore \frac{\partial z}{\partial x} = \frac{-\sec^2(x+y)}{\sec^2(y+z)} = -\frac{\cos^2(y+z)}{\cos^2(x+y)}$$

(c) $e^{xz}+xy=0$: Applying $\frac{\partial}{\partial x}$: $e^{xz}\left(x\frac{\partial z}{\partial x}+z\right) + x\cdot 0 + y = 0$

$$\therefore \frac{\partial z}{\partial x} = \frac{-y/e^{xz} - z}{x} \quad \text{etc.}$$

Applying $\frac{\partial}{\partial y}$: $e^{xz}\left[x\frac{\partial z}{\partial y}+0\right] + x + 0 = 0$

$$\therefore \frac{\partial z}{\partial y} = \frac{-x/e^{xz}}{x} = -\frac{1}{e^{xz}} = e^{-xz}$$