## Directional Derivatives

1. Find the directional derivative of the given function at the given point in the direction of the given vector **v**:

   (a) $f(x,y) = 3x - 4xy + 5y,\ P(1,2),\ v = \frac{1}{2}(i + \sqrt{3}j)$

   (b) $f(x,y) = xy,\ P(2,3),\ v = i + j$

   (c) $g(x,y) = \sqrt{x^2 + y^2},\ P(3,4),\ v = 3i - 4j$

   (d) $h(x,y) = e^x \sin y,\ P(1,\pi/2),\ v = -i$

2. Find the directional derivative of the function in the direction of $\mathbf{u} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$:

   $f(x,y) = x^2 + y^2,\ \theta = \pi/4$

3. Find the directional derivative of the given function in the direction from point P to point Q:

   $f(x,y) = x^2 + 4y^2,\ P(3,1),\ Q(1,-1)$

4. Find the gradient of the function and the maximum value of the directional derivative at the indicated point:

   (a) $h(x,y) = x \tan y,\ P(2,\pi/4)$

   (b) $g(x,y) = \ln\sqrt[3]{x^2 + y^2},\ P(1,2)$

   (c) $f(x,y,z) = xe^{yz},\ P(2,0,-4)$

5. Given the function

   $f(x,y) = 3 - \frac{x}{3} - \frac{y}{2}$:

   Find $D_u f(3,2)$, where $u = \frac{v}{\|v\|}$, where v is the vector from (1,2) to (-2,6)

6. Find a normal vector to the level curve $f(x,y)=x/(x^2+y^2)$ which is $f(x,y)=1/2$ at the point P(1, 1).

7. If the temperature at the point (x, y) on a metal plate is given by the function $T = x/(x^2+y^2)$, Find the direction from the point (3, 4) in which a bug would have to walk to cool off as rapidly as possible.

# Solutions for Directional Derivatives

$f(x,y) = 3x - 4xy + 5y$ ; $P(1,2)$.

1 (a) $\vec{V} = \frac{1}{2}i + \frac{\sqrt{3}}{2}j$ is already a unit vector.

$f_x = 3 - 4y = 3 - 4(2) = -5$, $f_y = -4x + 5 = -4(1) + 5 = 1$

so $\nabla f = (-5, 1)$, so $D_{\vec{V}} f = (-5,1)\cdot(\frac{1}{2}, \sqrt{3}/2) = \frac{\sqrt{3} - 5}{2}$

(b) $f(x,y) = xy$, $P(2,3)$. $\vec{u} = \frac{\vec{V}}{\|\vec{V}\|} = \frac{i + j}{\sqrt{2}} = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j$

so $D_{\vec{u}} f = \nabla f \cdot \vec{u}$. $f_x = 3$, $f_y = 2$

so $D_{\vec{u}} f = (3,2)\cdot(1/\sqrt{2}, 1/\sqrt{2}) = 5/\sqrt{2}$

(c) $g(x,y) = \sqrt{x^2 + y^2}$, $P(3,4)$, $\vec{u} = \frac{\vec{V}}{\|\vec{V}\|} = \frac{3}{5}i - \frac{4}{5}j$

$g_x = \frac{x}{\sqrt{x^2+y^2}} = \frac{3}{\sqrt{9+16}} = 3/5$

$g_y = \frac{y}{\sqrt{x^2+y^2}} = 4/5$, so $D_{\vec{u}} f = (\frac{3}{5}, \frac{4}{5})\cdot(\frac{3}{5}, -\frac{4}{5}) = -\frac{7}{25}$

(d) $h(x,y) = e^x \sin y$, $P(1, \pi/2)$, $\vec{V} = -i$ is a unit vector.

$h_x = e^x \sin y = e \sin \pi/2 = e$

$h_y = e^x \cos y = e \cos \pi/2 = 0$

so $D_{\vec{u}} f = (e, 0)\cdot(-1, 0) = -e$



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(2) $f(x,y) = x^2+y^2$, $\theta = \pi/4$

$\vec{u} = \cos\frac{\pi}{4} i + \sin \pi/4\, j = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$

$f_x = 2x$, $f_y = 2y$, so $D_{\vec{u}}f = (2x, 2y) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \dfrac{2x+2y}{\sqrt{2}}$

$$= \frac{2x+2y}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \sqrt{2}(x+y)$$

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(3) $f(x,y) = x^2+4y^2$, $P(3,1)$, $Q(1,-1)$, $\vec{PQ} = (-2,-2)$

$$\vec{u} = \frac{(-2,-2)}{\sqrt{4+4}} = \left(\frac{-2}{\sqrt{8}}, \frac{-2}{\sqrt{8}}\right) = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$$

$$D_u f = (2x, 8y) \cdot \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = \frac{-2x-8y}{\sqrt{2}} \text{ etc.}$$

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(4) $h(x,y) = x \tan y$, $P(2, \pi/4)$

(a) $h_x = \tan y$, $h_y = x \sec^2 y$ so at $P(2, \pi/4)$:

$h_x = \tan \pi/4 = 1$, $h_y = 2\sec^2\frac{\pi}{4} = 2 \cdot (\sqrt{2})^2 = 4$

$\nabla h = (1,4)$. By the theorem from lecture on max value of $D_{\vec{u}}f$, we know Max value of $D_{\vec{u}}h$ is $\|\nabla h\|$

$$= \sqrt{1^2+4^2} = \sqrt{17}$$

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(b) $h = \frac{1}{3}\ln(x^2+y^2)$, $h_x = \dfrac{2x}{3(x^2+y^2)} = \dfrac{2}{3(1+4)} = 2/15$

$h_y = \dfrac{2y}{3(x^2+y^2)} = \dfrac{2(2)}{3(1+4)} = 4/15$, $\nabla h = \left(\frac{2}{15}, \frac{4}{15}\right)$



Max value of $D_{\vec{u}}h = \|\nabla h\| = \sqrt{\left(\frac{2}{15}\right)^2 + \left(\frac{4}{15}\right)^2} = \dfrac{2\sqrt{5}}{15}$

(c) $f(x,y,z) = xe^{yz}$, $P(2,0,-4)$

$f_x = e^{yz} = e^0 = 1$, $f_y = xze^{yz} = -8e^0 = -8$,

$f_z = yxe^{yz} = 0$, so $\nabla f = (1,-8,0)$

Max value of $D_{\vec{u}}f = \|\nabla f\| = \|(1,-8,0)\| =$

$$\sqrt{1+64+0} = \sqrt{65}$$

5. $f(x,y) = 3 - \frac{x}{3} - \frac{y}{2}$, $\vec{v} = (-3,4)$, so

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{(-3,4)}{\|(-3,4)\|} = \frac{(-3,4)}{\sqrt{9+16}} = \left(-\frac{3}{5}, \frac{4}{5}\right)$$

$\nabla f = \left(-\frac{1}{3}, -\frac{1}{2}\right)$, $D_{\vec{u}}f = \left(-\frac{1}{3}, -\frac{1}{2}\right) \cdot \left(-\frac{3}{5}, \frac{4}{5}\right) = -1/5$

6. By a theorem from lecture, $\nabla f$ is normal to each level curve $f(x,y) = $ Constant

$$f_x = \frac{y^2 - x^2}{(x^2+y^2)^2} = 0 \text{ at } (1,1).$$

$$f_y = \frac{-2xy}{(x^2+y^2)^2} = -1/2 \text{ at } (1,1)$$

so $\nabla f = (0, -1/2)$ is normal to the level

Curve $\frac{1}{2} = \frac{x}{x^2+y^2}$ at $P(1,1)$



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(7) - The bug must walk in the direction $-\nabla T$ when leaving from the point $(3,4)$.

$$T_x = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{16-9}{25^2} = 7/625$$

$$T_y = \frac{-2xy}{(x^2+y^2)^2} = \frac{-2(3)(4)}{25^2} = -24/625$$

$$-\nabla T = \left(-7/625,\ 24/625\right)$$