Tangent Planes and Normal Lines

1. Find a unit normal vector to the surface at the indicated point:

   (a) $x + y + z = 4,\ P(2,0,2)$
   (b) $z = \sqrt{x^2 + y^2},\ P(3,4,5)$
   (c) $x^2 y^4 - z = 0,\ P(1,2,16)$
   (d) $z - x\sin y = 4,\ P(6,\pi/6,7)$

2. Find an equation of the tangent plane to the surface at the indicated point:

   (a) $z = 25 - x^2 - y^2,\ P(3,1,15)$
   (b) $f(x,y) = y/x,\ P(1,2,2)$
   (c) $g(x,y) = x^2 - y^2,\ P(5,4,9)$
   (d) $z = e^x(\sin y + 1),\ P(0,\pi/2,2)$
   (e) $h(x,y) = \ln\sqrt{x^2 + y^2},\ P(3,4,\ln 5)$
   (f) $x^2 + 4y^2 + z^2 = 36,\ P(2,-2,4)$
   (g) $xy^2 + 3x - z^2 = 4,\ P(2,1,-1)$

3. Find the equation of the normal line to the surface at the indicated point:

   (a) $x^2 + y^2 + z = 9,\ P(1,2,4)$
   (b) $xy - z = 0,\ P(-2,-3,6)$
   (c) $xyz = 10,\ P(1,2,5)$

4. Show that any tangent plane to the cone $z^2 = (ax)^2 + (by)^2$ passes through the origin.

*5. Suppose that f is any differentiable function of a single variable and suppose that a surface is defined by $z = xf(y/x)$. Show that the tangent plane at every point of this surface passes through the origin.

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Solutions for Tangent Planes and Normal Lines

(a) $F(x,y,z) = x+y+z-4$, $F_x = 1$, $F_y = 1$, $F_z = 1$
$\nabla F = (1,1,1)$ is normal to the Given Surface at every point (note the given Surface is a plane) so $\frac{\nabla F}{\|\nabla F\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is a unit normal.

(b) $F = \sqrt{x^2+y^2} - z$, $P(3,4,5)$, $F_x = \frac{x}{\sqrt{x^2+y^2}} = \frac{3}{\sqrt{3^2+4^2}} = \frac{3}{5}$
$F_y = \frac{y}{\sqrt{x^2+y^2}} = \frac{4}{\sqrt{3^2+4^2}} = 4/5$, $F_z = -1$, so
$\nabla F = \left(\frac{3}{5}, \frac{4}{5}, -1\right)$, $\frac{\nabla F}{\|\nabla F\|} = \left(\frac{3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ is a unit normal.

(c) $F = x^2y^4 - z$, $P(1,2,16)$, $F_x = 2xy^4 = 2(1)\cdot 2^4 = 32$,
$F_y = 4x^2y^3 = 4(1)^2\cdot 2^3 = 32$, $F_z = -1$, so $\nabla F = (32,32,-1)$
and so $\frac{\nabla F}{\|\nabla F\|} = \left(\frac{32}{\sqrt{2049}}, \frac{32}{\sqrt{2049}}, \frac{-1}{\sqrt{2049}}\right)$ is a unit normal.

(d) $F = z - x\sin y$, $P(6, \pi/6, 7)$, $F_x = -\sin y = -\sin\frac{\pi}{6} = -1/2$,
$F_y = -x\cos y = -6\frac{\sqrt{3}}{2} = -3\sqrt{3}$, $F_z = 1$, $\nabla F = \left(-\frac{1}{2}, -3\sqrt{3}, 1\right)$, so
$\frac{\nabla F}{\|\nabla F\|} = \frac{2}{\sqrt{113}}\left(\frac{-1}{2}, -3\sqrt{3}, 1\right)$ etc.

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(a) $F = z + x^2 + y^2 - 25$, $P(3,1,15)$, $F_x = 2x = 2(3) = 6$
$F_y = 2y = 2(1) = 2$, $F_z = 1$. So tan plane is:
$6(x-3) + 2(y-1) + z - 15 = 0$ or
$6x + 2y + z = 35$

(b) $F = z - \frac{y}{x}$; $P(1,2,2)$, $F_x = y/x^2 = 2/1^2 = 2$,
$F_y = -1/x = -1/1 = -1$, $F_z = 1$, so tan plane is:
$2(x-1) + -1(y-2) + 1(z-2) = 0$ or $2x - y + z = 2$

(c) $F = z + y^2 - x^2$, $P(5,4,9)$, $F_x = -2x = -2(5) = -10$,
$F_y = 2y = 2(4) = 8$, $F_z = 1$, so tan plane is:
$-10(x-5) + 8(y-4) + 1(z-9) = 0$, or $10x - 8y - z = 9$.

(d) $F = z - e^x(\sin y + 1)$, $P(0, \pi/2, 2)$, $F_x = -e^x(\sin y + 1) =$
$-e^0(\sin\frac{\pi}{2} + 1) = -2$, $F_y = -e^x(\cos y) = -e^0 \cos \pi/2 = 0$, $F_z = 1$,
so Tan plane is: $-2(x-0) + 0(y - \pi/2) + 1(z-2) = 0$ or
$2x - z = -2$.

(e) $F = \frac{1}{2}\ln(x^2+y^2) - z$, $F_x = x/(x^2+y^2) = 3/(9+16) = 3/25$,
$F_y = y/(x^2+y^2) = 4/(9+16) = 4/25$, $F_z = -1$, so tan plane
is: $\frac{3}{25}(x-3) + \frac{4}{25}(y-4) + -1(z - \ln(5)) = 0$ or
$3x + 4y - 25z = 25 - 25\ln 5$

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(f) $F = x^2 + 4y^2 + z^2 - 36$, $p(2,-2,4)$; $F_x = 2x = 2(2) = 4$,
$F_y = 8y = 8(-2) = -16$, $F_z = 2z = 2(4) = 8$, so tan plane is:
$4(x-2) + -16(y+2) + 8(z-4) = 0$ or $x - 4y + 2z = 18$.

(g) $F = xy^2 + 3x - z^2 - 4$, $p(2,1,-1)$, $F_x = y^2 + 3 = 1 + 3 = 4$,
$F_y = 2xy = 2(2)(1) = 4$, $F_z = -2z = -2(-1) = 2$, so tan plane is: $4(x-2) + 4(y-1) + 2(z+1) = 0$ or
$2x + 2y + z = 5$

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③ (a) $F = x^2 + y^2 + z - 9$, $p(1,2,4)$, $F_x = 2x = 2(1) = 2$,
$F_y = 2y = 2(2) = 4$, $F_z = 1$, so $\nabla F = (2,4,1)$ is normal to given surface, hence parallel to desired line, so line is:

$$\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-4}{1}$$

(b) $F = xy - z$, $p(-2,-3,6)$. $F_x = y = -3$, $F_y = x = -2$, $F_z = -1$
so line is: $\frac{x+2}{-3} = \frac{y+3}{-2} = \frac{z-6}{-1}$ or

$$\frac{x+2}{3} = \frac{y+3}{2} = \frac{z-6}{1}$$

(c) $F = xyz - 10$, $p(1,2,5)$, $F_x = yz = 10$, $F_y = xz = 5$,
$F_z = xy = 2$, so normal line is: $\frac{x-1}{10} = \frac{y-2}{5} = \frac{z-5}{2}$

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(4) Consider the tangent plane to the given Cone at $P_0(x_0, y_0, z_0)$ on Cone. Then $z_0^2 = a^2x_0^2 + b^2y_0^2$. Let $F = a^2x^2 + b^2y^2 - z^2$, so

$$F_x = 2a^2x = 2a^2x_0;\ F_y = 2b^2y = 2b^2y_0,\ F_z = -2z = -2z_0.$$

Tan plane to Cone at P_0 is: $2a^2x_0(x - x_0) + 2b^2y_0(y - y_0) - 2z_0(z - z_0) = 0$

Now note $(0,0,0)$ satisfies the above tan plane equation:

$$-2a^2x_0^2 + -2b^2y_0^2 + 2z_0^2 = -2[\underbrace{(ax_0)^2 + (by_0)^2 - z_0^2}_{0}] = 0$$

∴ The tan plane to the Cone at an arbitrary point (x_0, y_0, z_0) passes through the origin.

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(5) Let $F = xf(y/x) - z$ and Consider the tan. plane to given Surface at $P_0(x_0, y_0, z_0)$. $F_x = xf'(\frac{y}{x}) \cdot (\frac{-1}{x^2})y + f(\frac{y}{x}) = -\frac{y_0}{x_0}f'(y_0/x_0) + f(y_0/x_0)$,

$$F_y = x \cdot f'(\tfrac{y}{x}) \cdot \tfrac{1}{x} = f'(y/x) = f'(y_0/x_0),\ F_z = -1, \text{ so}$$

tan plane is: $\left[-\frac{y_0}{x_0}f'(\frac{y_0}{x_0}) + f(\frac{y_0}{x_0})\right][x - x_0] + f'(\frac{y_0}{x_0})[y - y_0] - [z - z_0] = 0$

at $(0,0,0)$ we have: $y_0f'(\frac{y_0}{x_0}) - x_0f(\frac{y_0}{x_0}) + f'(\frac{y_0}{x_0})(-y_0) + z_0 =$

$$\cancel{y_0 f'(\tfrac{y_0}{x_0})} - \cancel{x_0 f(\tfrac{y_0}{x_0})} + \cancel{f'(\tfrac{y_0}{x_0})(-y_0)} + x_0 f(\tfrac{y_0}{x_0}) = 0$$

∴ $(0,0,0)$ lies on every tan. plane to given Surface.