Dazzling Double Integrals

Evaluate:

1. $$\int_0^1 \int_0^2 (x+y)\,dy\,dx$$

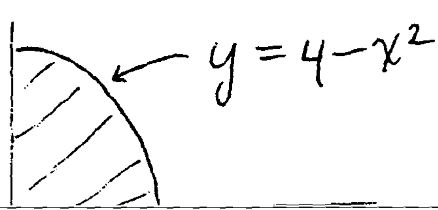
2. $$\int_1^2 \int_0^4 (x^2-2y^2+1)\,dx\,dy$$

3. $$\int_0^1 \int_0^{\sqrt{1-y^2}} (x+y)\,dx\,dy$$

4. $$\int_0^2 \int_{3y^2-6y}^{2y-y^2} 3y\,dx\,dy$$

5. $$\int_0^2 \int_0^{\sqrt{4-y^2}} \frac{2}{\sqrt{4-y^2}}\,dx\,dy$$

6. Use a double integral to find the area shown:



(7) Use double integrals to find the area of the region bounded by the graphs: $\sqrt{x} + \sqrt{y} = 2, x=0, y=0$.

(8) use double integrals to find the area bounded by $2x-3y=0$, $x+y=5$, $y=0$.

(9) Find (via double integrals) the area bounded by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(10) Evaluate the integral — you may need to switch the order of integration:

(a) $\displaystyle\int_0^2 \int_x^2 x\sqrt{1+y^3}\, dy\, dx$

(b) $\displaystyle\int_0^1 \int_y^1 \sin x^2\, dx\, dy$

Solutions to Dazzling Double Integrals

(1) $\displaystyle\int_0^1\int_0^2 (x+y)\,dy\,dx = \int_0^1 \Big[xy + y^2/2\Big]_{y=0}^{2} dx =$

$\displaystyle\int_0^1 (2x+2)\,dx = x^2+2x\Big]_{x=0}^{1} = 3$

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(2) $\displaystyle\int_1^2\int_0^4 (x^2-2y^2+1)\,dx\,dy = \int_1^2 \Big[\frac{x^3}{3} - 2y^2x + x\Big]_{x=0}^{4} dy$

$\displaystyle= \int_1^2 \left(\frac{64}{3} - 8y^2 + 4\right) dy = \int_1^2 \left(\frac{76}{3} - 8y^2\right) dy$

$\displaystyle= \frac{76y}{3} - \frac{8}{3}y^3\Big]_{y=1}^{2} = \left(\frac{152}{3} - \frac{64}{3}\right) - \left(\frac{76}{3} - \frac{8}{3}\right)$

$= 20/3$

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(4) $\displaystyle\int_0^2\int_{3y^2-6y}^{2y-y^2} 3y\,dx\,dy = \int_0^2 \Big[3xy\Big]_{x=3y^2-6y}^{2y-y^2} dy$

$\displaystyle= \int_0^2 3y\{(2y-y^2)-(3y^2-6y)\}\,dy = \int_0^2 3y(8y-4y^2)\,dy$

$\cancel{2y} - y^2 - 3y^2 + \cancel{6y}$

$\displaystyle= \int_0^2 (24y^2-12y^3)\,dy = 8y^3 - 3y^4\Big]_{y=0}^{2} = 16$

(3) $$\int_0^1\int_0^{\sqrt{1-y^2}}(x+y)\,dx\,dy = \int_0^1 \left[\frac{x^2}{2}+xy\right]_{x=0}^{\sqrt{1-y^2}} dy =$$

$$\int_0^1\left(\frac{1-y^2}{2}+y\sqrt{1-y^2}\right)dy = \frac{1}{2}y-\frac{y^3}{6}+-\frac{1}{3}(1-y^2)^{3/2}\Big]_{y=0}^{1}$$

$$=\left[\frac{1}{2}-\frac{1}{6}-\frac{1}{3}(0)\right]-\left[-\frac{1}{3}\right]=2/3$$

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(5) $$\int_0^2\int_0^{\sqrt{4-y^2}}\frac{2}{\sqrt{4-y^2}}\,dx\,dy = \int_0^2\left[\frac{2x}{\sqrt{4-y^2}}\right]_{x=0}^{\sqrt{4-y^2}}dy$$

$$=\int_0^2 2\,dy = 2y\Big]_{y=0}^{2} = 4$$

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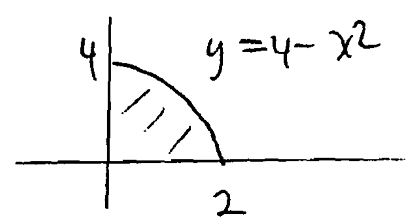
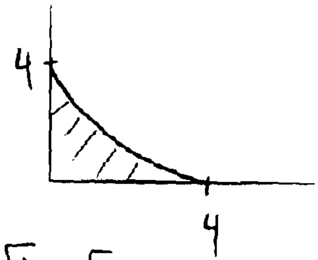
(6) $$A = \int_{x=0}^{2}\int_{y=0}^{4-x^2} dy\,dx$$



$$=\int_0^2(4-x^2)\,dx = 4x-\frac{x^3}{3}\Big]_0^2 = 8-\frac{8}{3} = 16/3$$

10

(7)



$$A = \int_{x=0}^{4}\int_{y=0}^{4-4\sqrt{x}+x} dy\,dx = \cdots = \boxed{8/3}$$

$$\sqrt{x}+\sqrt{y} = 2$$

$$\sqrt{y} = 2-\sqrt{x}$$

$$y = (2-\sqrt{x})^2 = 4 - 4\sqrt{x} + x$$

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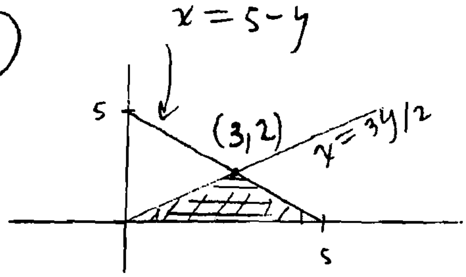
(8)



Area of shaded region

$$= \int_{y=0}^{2}\int_{x=\frac{3y}{2}}^{5-y} dx\,dy = \cdots = \boxed{5}$$

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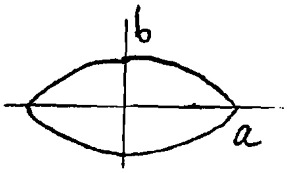
(9)



In QI, $y = \frac{b}{a}\sqrt{a^2-x^2}$

total Area = 4 · (first Quadrant Area)

$$= 4\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dy\,dx = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

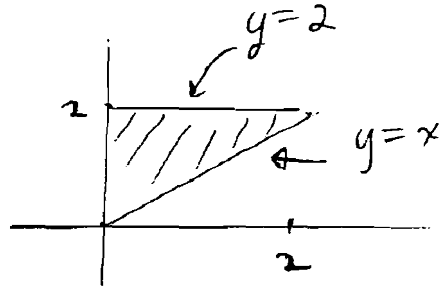
$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$ Let $x = a\sin\theta$, so $dx = a\cos\theta\,d\theta$

$$= \frac{4b}{a}\int_{\theta=0}^{\pi/2} \sqrt{a^2(1-\cos^2\theta)} \cdot a\cos\theta\,d\theta = 4ab\int_{\theta=0}^{\pi/2} \cos^2\theta\,d\theta$$

$$= 4ab\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{4ab}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \boxed{\pi ab}$$

10 (a)

Region of integration is :

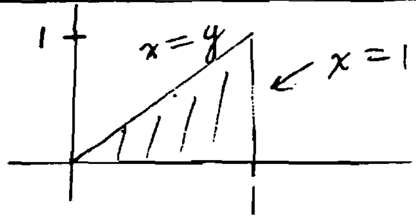


so $$\int_0^2 \int_x^2 x\sqrt{1+y^3}\, dy\, dx = \int_{y=0}^{2} \int_{x=0}^{y} x\sqrt{1+y^3}\, dx\, dy$$

$$= \int_{y=0}^{2} \left[\sqrt{1+y^3}\,\frac{x^2}{2}\right]_{x=0}^{y} dy = \int_{y=0}^{2} \frac{y^2}{2}\sqrt{1+y^3}\, dy$$

$$= \cdots = 26/9$$

(b) Region of integration is



$$\int_0^1 \int_y^1 \sin x^2\, dx\, dy = \int_{x=0}^{1} \left[\int_{y=0}^{x} \sin x^2\, dy\right] dx$$

$$= \int_0^1 x\sin x^2\, dx = -\frac{1}{2}(\cos 1 - 1) = \frac{1}{2}(1 - \cos 1)$$