More vector Problems – dot product

1. For the following vectors **u** and v, find the following quantities:

   $(a)\ \boldsymbol{u}\bullet\boldsymbol{v} \quad (b)\ \boldsymbol{v}\bullet\boldsymbol{v} \quad (c)\ \|\boldsymbol{u}\|^2 \quad (d)\ (\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{v} \quad (e)\ \boldsymbol{u}\bullet(2\boldsymbol{v})$

   (i) **u** = (1, -3, 5) and **v** = (-2, -3, 4)

   (ii) **u** = (1, 0, -2) and **v** = (0, 3, -5)

2. Find the inner product (dot product) of **u** and **v** if the magnitudes of **u** and **v** are, respectively, 10 and 6 and the angle acute angle between them is 60 degrees.

3. Find the angle between the vectors **u** = 3i – 2j + 5k and **v** = 5i –3j+7k.

4. Determine whether u and v are orthogonal, parallel or neither:
   u = (4, 3) and **v** = (1/2, -2/3).

5. Find the angle between a cube's main diagonal and one of its edges. Also find the angle between a cubes main diagonal and the diagonal of its square base.

6. Find two vectors in opposite directions that are orthogonal to **v** = -8i+3j.

7. Find the work done in moving an object from P to Q if the magnitude and direction of the force are given by the vector **v**:
   P(0, 0, 0), Q(4, 7, 5), **v**= (1, 4, 8).

8**. Prove the **Cauchy-Schwarz Inequality**:

   $|\boldsymbol{u}\bullet\boldsymbol{v}| \leq \|\boldsymbol{u}\|\,\|\boldsymbol{v}\|$

   [Hint: This inequality requires a rather clever idea. Consider the real function

   $f(t) = \|t\boldsymbol{u}+\boldsymbol{v}\|^2$, for all real numbers t

   Note that f(t) is never less than zero and can also be expressed using the dot product – do that and then apply basic properties of the dot product listed in your text in order to expand the resulting expression...now consider that expression as a quadratic function (parabola) in the variable t....now its up to you to make the clever observation to finish the proof!].

Dot Product Solutions

1 (i) $\vec{u} = (1,-3,5)$, $\vec{v} = (-2,-3,4)$

$$\vec{u}\cdot\vec{v} = 1(-2) + (-3)(-3) + 5(4) = 27$$

$$\vec{v}\cdot\vec{v} = (-2)^2 + (-3)^2 + 4^2 = 29$$

$$\|\vec{u}\|^2 = \vec{u}\cdot\vec{u} = 1^2 + (-3)^2 + 5^2 = 35$$

$$\vec{u}\cdot 2\vec{v} = 2(\vec{u}\cdot\vec{v}) = 2(27) = 54$$

$$(\vec{u}\cdot\vec{v})\vec{v} = 27\vec{v} = (-54,-81,108)$$

(ii) $\vec{u} = (1,0,-2)$, $\vec{v} = (0,3,-5)$

$$\vec{u}\cdot\vec{v} = 1(0) + 0(3) + (-2)(-5) = 10$$

$$\vec{v}\cdot\vec{v} = 0^2 + 3^2 + (-5)^2 = 34$$

$$\|\vec{u}\|^2 = \vec{u}\cdot\vec{u} = 1^2 + 0^2 + (-2)^2 = 5$$

$$(\vec{u}\cdot\vec{v})\vec{v} = 10\vec{v} = (0,30,-50)$$

$$\vec{u}\cdot 2\vec{v} = 2(\vec{u}\cdot\vec{v}) = 2(10) = 20$$

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2 $$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta = (10)(6)\cos 60^\circ = 1/2$$

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3 $$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

$$3(5) + (-2)(-3) + 5(7) = \sqrt{3^2+(-2)^2+5^2}\sqrt{5^2+(-3)^2+7^2}\cos\theta$$

$$56 = \sqrt{38}\sqrt{83}\cos\theta$$

$$\frac{56}{\sqrt{3154}} = \cos\theta$$

$$\theta \approx 4.33^\circ$$

④ $\vec{u}\cdot\vec{v} = 4(1/2) + 3(-\frac{2}{3}) = 0$

$\therefore$ $\vec{u}$ and $\vec{v}$ are orthogonal

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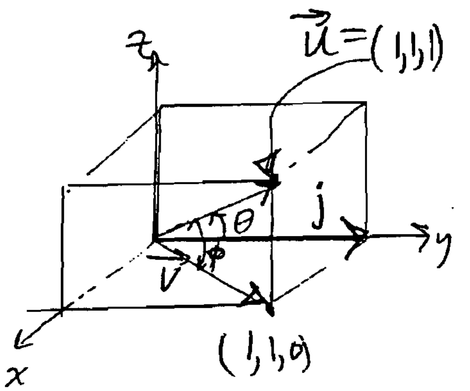
⑤



To find angle between main diagonal ~~and y=Ax is~~ ;

$\vec{u}\cdot j = (1,1,1)\cdot(0,1,0) = 1$

Also $\vec{u}\cdot j = \|\vec{u}\|\|j\|\cos\theta = \sqrt{3}\cos\theta$

$\therefore 1 = \sqrt{3}\cos\theta$

$\theta \approx 54.74^\circ$

To find angle between $\vec{u}$ and $\vec{v}$;

$\vec{u}\cdot\vec{v} = (1,1,1)\cdot(1,1,0) = 2$

$\vec{u}\cdot\vec{v} = \|(1,1,1)\|\ \|(1,1,0)\|\cos\phi$

$2 = \sqrt{3}\sqrt{2}\cos\phi$

$\phi \approx 35.26^\circ$

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⑥ one vector orthogonal to $\vec{v} = -8i + 3j$ is

$\vec{w_1} = 3i + 8j$ (since $\vec{v}\cdot\vec{w_1} = 0$), ~~so~~ the one in the

opposite direction is $\vec{w_2} = -\vec{w_1} = -3i - 8j$

(7) work $= \vec{V}\cdot\vec{PQ} = (1,4,8)\cdot(4,7,5) = 4+28+40$
$= 72$

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(8)** $f(t) = \|t\vec{u}+\vec{v}\|^2 = (t\vec{u}+v)\cdot(t\vec{u}+v)$

$= t^2\vec{u}\cdot\vec{u} + 2t\,\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$

$= (\vec{u}\cdot\vec{u})t^2 + 2(\vec{u}\cdot\vec{v})\,t + \vec{v}\cdot\vec{v}$

Since $\|t\vec{u}+\vec{v}\|^2 \geq 0$, $\underbrace{(\vec{u}\cdot\vec{u})}_{A}t^2 + \underbrace{2(\vec{u}\cdot\vec{v})}_{B}\,t + \underbrace{\vec{v}\cdot\vec{v}}_{C} \geq 0$
for all values of t.

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Now for any quadratic function $y = At^2 + Bt + C$ lying above the t-axis, the discriminant $B^2 - 4AC < 0$ (i.e. any attempt to compute the t-intercept, by letting $y=0$ must necessarily yield no solution!).

$\therefore\ 4(\vec{u}\cdot\vec{v})^2 - 4(\vec{u}\cdot\vec{u})(\vec{v}\cdot\vec{v}) < 0$

$\therefore\ 4(\vec{u}\cdot\vec{v})^2 - 4\|\vec{u}\|^2\|\vec{v}\|^2 < 0$

$\therefore\ (\vec{u}\cdot\vec{v})^2 \leq \|\vec{u}\|^2\|\vec{v}\|^2$

$\therefore\ |\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\,\|\vec{v}\|$