Cool Cross Product Problems

1.

Find $\boldsymbol{u} \times \boldsymbol{v}$ and show that its orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$.

$\mathbf{u} = (2, -3, 1)$ and $\mathbf{v} = (1, -2, 1)$

2. repeat #1 if $\mathbf{u} = \mathbf{i}+\mathbf{j}+\mathbf{k}$ **and** $\mathbf{v} = 2\mathbf{i}+\mathbf{j}-\mathbf{k}$

3. Compute the area of the following parallelogram having the given vectors as adjacent sides: $\mathbf{u} = i+j+k$ and $\mathbf{v} = j+k$.

4. Verify the following points are the vertices of a parallelogram, and find its area: (1,1,1), (2,3,4), (6,5,2), (7,7,5).

5. Find the area of the triangle with vertices (2, -3, 4), (0,1,2), (-1,2,0).

6. Find the volume of the parallelepiped having adjacent edges **u**, **v**, **w** where $\mathbf{u} = \mathbf{i}+\mathbf{j}$, $\mathbf{v} = \mathbf{j}+\mathbf{k}$, and $\mathbf{w} = \mathbf{i}+\mathbf{k}$.

7. Find the volume of the parallelepiped with the vertices (0,0,0), (3,0,0), (0,5,1), (3,5,1), (2,0,5), (5,0,5), (2,5,6), (5,5,6).

8. A child applies the brakes on a bicycle by applying a downward force of 20 lbs. on a pedal which is attached to the crank (a lever arm) when the crank makes a 40 degree angle with the horizontal. Find the torque at the crank's pivot point if the crank is 6 inches long.

9.

Consider the vectors $\boldsymbol{u} = (\cos\alpha, \sin\alpha, 0)$ and $\boldsymbol{v} = (\cos\beta, \sin\beta, 0)$, where $\alpha > \beta$. Find the cross product of the vectors and use the result to prove $\sin(\alpha - \beta) = \sin\alpha\,\cos\beta - \cos\alpha\,\sin\beta$.

Cool Cross Product Problems

(1) $\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 2 & -3 & 1 \\ 1 & -2 & 1 \end{vmatrix} = i(-3+2) - j(2-1) + k(-4+3)$

$= -i - j - k$

note $\vec{u} \cdot (-1,-1,-1) = (2,-3,1) \cdot (-1,-1,-1) = -2+3-1 = 0$

and $\vec{v} \cdot (-1,-1,-1) = (1,-2,1) \cdot (-1,-1,-1) = -1+2-1 = 0$

so this verifies $(-1,-1,-1)$ is orthog. to each of $\vec{u}$ and $\vec{v}$.

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(2) $\vec{u} \times \vec{v} = (-2,3,-1)$, etc.

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(3) Area $= \|\vec{u} \times \vec{v}\| = \|(0,-1,1)\| = \sqrt{0^2+(-1)^2+1^2} = \sqrt{2}$

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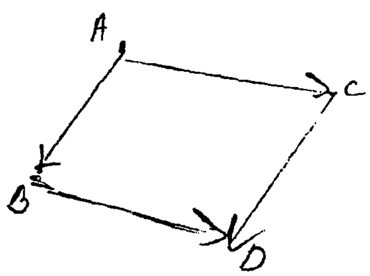
(4)



$\vec{AB} = (1,2,3) = \vec{CD}$

$\vec{AC} = (5,4,1) = \vec{BD}$

Area $= \|\vec{AB} \times \vec{AC}\| = \|(-10,14,-6)\|$

$= \sqrt{100+196+36} = \sqrt{332}$

$\doteq 18.22$

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(5) $A(2,-3,4)$, $B(0,1,2)$, $C(-1,2,0)$

$\vec{BA} = (2,-4,2)$, $\vec{BC} = (-1,1,-2)$

Area $= \frac{1}{2}\|\vec{BA} \times \vec{BC}\| = \frac{1}{2}\|(6,2,-2)\| =$

$\frac{1}{2}\sqrt{36+4+4} = \frac{1}{2}\sqrt{44} = \sqrt{11}$

(OVER→)

(6) $\text{Volume} = |(\vec{u} \times \vec{v}) \cdot \vec{w}|$

$= |(1,-1,1) \cdot (1,0,1)| = 2$

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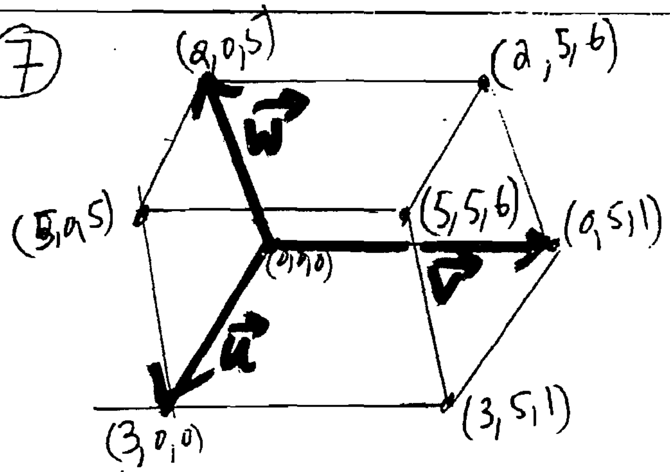
(7)



$\text{volume} = |(\vec{u} \times \vec{v}) \cdot \vec{w}|$

$= |(0,-3,15) \cdot (2,0,5)|$

$= 75$

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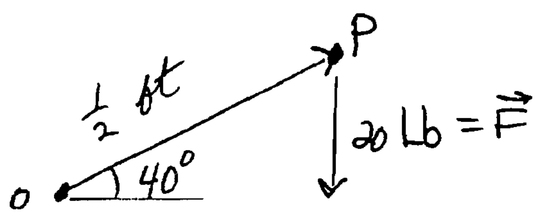
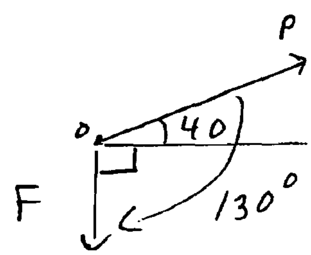
(8)



Torque at O $= \|\vec{F} \times \vec{OP}\| = \|\vec{F}\|\,\|\vec{OP}\| \sin 130^\circ$

since angle between $\vec{OP}$ and $\vec{F}$ is 130°



so Torque $= \underset{\text{Lb}}{(20)}(1/2\text{ ft})\sin 130^\circ \doteq 7.66\text{ ft}\cdot\text{Lb}$

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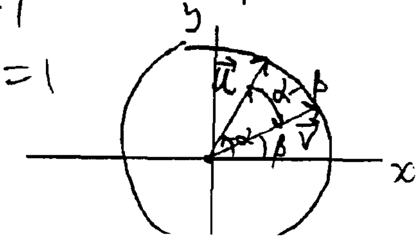
(9) $\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \end{vmatrix} = k(\cos\alpha\sin\beta - \cos\beta\sin\alpha)$, so

$\|\vec{u} \times \vec{v}\| = |\cos\alpha\sin\beta - \cos\beta\sin\alpha| = \sin\alpha\cos\beta - \cos\beta\sin\alpha$

$=1$

$1=1$



Also $\|\vec{u} \times \vec{v}\| = \underbrace{\|\vec{u}\|}_{1}\underbrace{\|\vec{v}\|}_{1}\sin(\alpha-\beta) = \sin(\alpha-\beta)$

angle between vectors.

$\therefore \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

right-handed