Vectors-velocity and Acceleration

1. A baseball, hit 3 feet above the ground, leaves the bat at an angle of 45 degrees and is caught by an outfielder 300 ft. from home plate. What is the initial speed of the ball and how high does it rise if it is caught 3 ft. above the ground?

2. The quarterback of a football team releases a pass at a height of 7 ft. above the playing field, and the football is caught by a receiver 30 yards directly downfield at a height of 4 feet. The pass is released at an angle of 35 degrees with the horizontal. Answer the following:

   (a) Find the speed of the football when it is released.

   (b) Find the maximum height of the football.

   (c) Find the time the receiver has to reach the proper position after the quarterback releases the football.

3. A shot fired from a gun with a muzzle velocity of 1200 ft/sec is to hit a target 3000 ft away. Determine the minimum angle of elevation of the gun.

4. Consider the motion of a point on the circumference of a circle rolling along on a flat surface. As it rolls, it generates the cycloid:

   $\mathbf{r}(t) = b(wt - \sin wt)\mathbf{i} + b(1 - \cos wt)\mathbf{j}$, where w is the angular velocity of the circle. of radius b

   (a) Find the velocity and acceleration vectors of the particle. Use the results to determine the times at which the speed of the moving point will be zero. When will the speed be maximized?

   (b) Find the maximum speed of a point on the circumference of an automobile tire of radius 1 foot when the automobile is traveling at 55 miles/hr. Compare this speed with the speed of the automobile.

5. The path of a shot thrown at an angle theta is given by :

$$r(t) = (v_0 \cos\theta)t\, i + [h + (v_0 \sin\theta)t - \frac{1}{2}gt^2]\, j$$

Where v_0 is the initial speed and h is the initial height and t is time in seconds and g is the acceleration due to gravity. Verify that the shot will remain in the air for

$$t = \frac{v_0 \sin\theta + \sqrt{v_0^2 \sin^2\theta + 2gh}}{g} \text{ seconds}$$

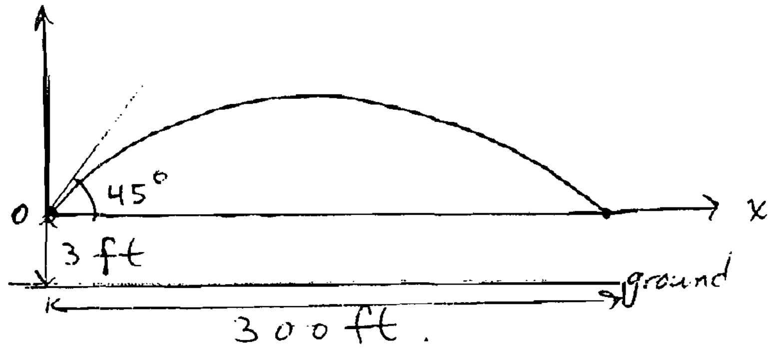
and will travel a total horizontal distance of

$$\frac{v_0^2 \cos\theta}{g}\left[\sin\theta + \sqrt{\sin^2\theta + \frac{2gh}{v_0^2}}\right] \text{ feet.}$$

6. Prove that if an object is traveling at constant speed, its velocity and acceleration vectors are always orthogonal.

7. Prove that an object moving in a straight line at a constant speed has an acceleration of zero.

Vectors – Velocity and Accel. – Solutions



$$x(t) = v_0 \cos\theta\, t = \frac{v_0 t}{\sqrt{2}}$$

$$y(t) = v_0 \sin\theta\, t - 16t^2 = \frac{v_0 t}{\sqrt{2}} - 16t^2$$

time taken to travel 300 ft horizontally, t, satisfies

$$300 = \frac{v_0 t}{\sqrt{2}} \quad \therefore\ t = \frac{300\sqrt{2}}{v_0}, \text{ so}$$

$$0 = y\left(\frac{300\sqrt{2}}{v_0}\right) = \frac{v_0}{\sqrt{2}}\left(\frac{300\sqrt{2}}{v_0}\right) - 16\left(\frac{300\sqrt{2}}{v_0}\right)^2$$

$$\Longrightarrow \underline{v_0 = 40\sqrt{6}\ \text{ft/sec}}, \text{ and } t = \frac{300\sqrt{2}}{40\sqrt{6}} = \frac{5\sqrt{3}}{2}$$

Max height occurs when $y'(t) = 0$: $\dfrac{v_0}{\sqrt{2}} - 32t = 0$

$$\Longrightarrow t = \frac{v_0}{32\sqrt{2}} = \frac{40\sqrt{6}}{32\sqrt{2}} = \frac{5\sqrt{3}}{4} \text{ sec}$$

At this time, $y(t) = y\left(\frac{5\sqrt{3}}{4}\right) = \frac{40\sqrt{6}}{\sqrt{2}}\left(\frac{5\sqrt{3}}{4}\right) - 16\left(\frac{5\sqrt{3}}{4}\right)^2 = 75$ ft.

i.e. 78 ft above ground level.

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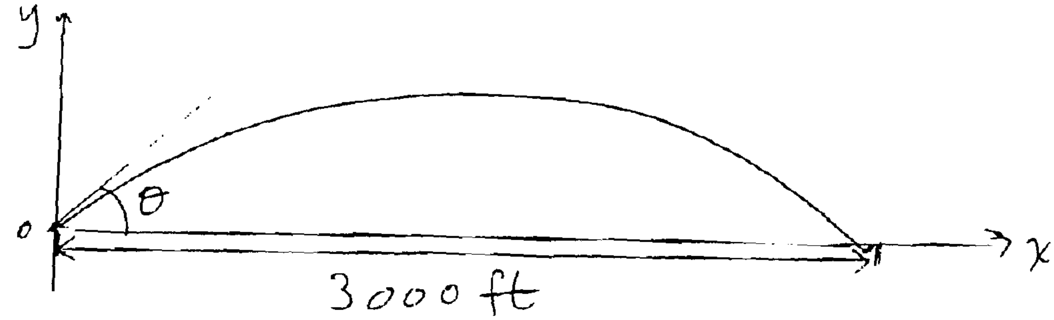

$$x(t) = 1200\cos\theta\, t, \quad y(t) = -16t^2 + 1200\sin\theta\, t$$

$$1200\cos\theta\, t = 3000 \implies t = \frac{5}{2\cos\theta}$$

$$y(t) = -16\left(\frac{5}{2\cos\theta}\right)^2 + 1200\sin\theta\left(\frac{5}{2\cos\theta}\right) = 0$$

when $-\dfrac{100}{\cos^2\theta} + \dfrac{3000\sin\theta}{\cos\theta} = 0$

$$-100 + 3000\sin\theta\cos\theta = 0$$

$$\sin\theta\cos\theta = \frac{1}{30}$$

$$\frac{\sin 2\theta}{2} = \frac{1}{30}$$

$$\sin 2\theta = \frac{1}{15} \implies 2\theta = \sin^{-1}(1/15) \doteq 3.8^\circ$$

$$\therefore \theta \doteq 1.9^\circ$$

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(4) (a) $\vec{v} = \dfrac{d\vec{r}}{dt} = b(w - w\cos wt)i + bw\sin wt\, j$

$$\vec{a} = \frac{d\vec{v}}{dt} = bw^2\sin wt\, i + bw^2\cos wt\, j$$

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(5) $y(t) = 0$ when $-\frac{1}{2}gt^2 + v_0 \sin\theta\, t + h = 0$

$$gt^2 + (-2v_0\sin\theta)t - 2h = 0$$

$$t = \frac{2v_0\sin\theta \pm \sqrt{4v_0^2\sin^2\theta + 8gh}}{2g}$$

$$= \frac{v_0\sin\theta + \sqrt{v_0^2\sin^2\theta + 2gh}}{g}$$

("$-$" is discarded since $\sqrt{v_0^2\sin^2\theta + 2gh} > v_0\sin\theta$ and so "$-$" would produce a negative value for t).

At this time $x(t) = v_0\cos\theta\, t = v_0\cos\theta\left[\frac{v_0\sin\theta + \sqrt{v_0^2\sin^2\theta + 2gh}}{g}\right]$

$$= \frac{v_0^2\cos\theta}{g}\left[\sin\theta + \sqrt{\sin^2\theta + \frac{2gh}{v_0}}\right]$$

(upon factoring out v_0^2 - i.e.

$\sqrt{v_0^2\sin^2\theta + 2gh} = \sqrt{v_0^2\left[\sin^2\theta + \frac{2gh}{v_0^2}\right]} = v_0\sqrt{\sin^2\theta + \frac{2gh}{v_0^2}}$)

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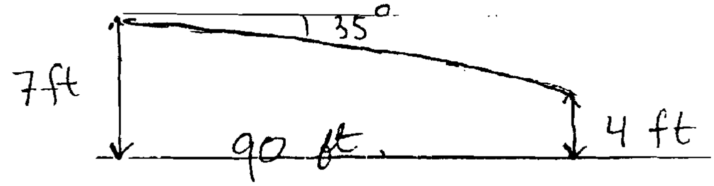
2 (a)



$x(t) = V_0 \cos 35° t$, $y(t) = V_0 \sin 35° t - 16t^2 + 7$

where $y(t)$ = number of ft. above ground at time t sec.

$V_0 \cos 35° t = 90 \Rightarrow t = \dfrac{90}{V_0 \cos 35°}$. At this time

$$4 = y\left(\frac{90}{V_0 \cos 35°}\right) = V_0 \sin 35° \left(\frac{90}{V_0 \cos 35°}\right) - 16\left(\frac{90}{V_0 \cos 35°}\right)^2 + 7$$

$\Rightarrow V_0 \doteq 54.1$ ft/sec.

(b) Max height occurs when $y'(t) = 0$: $y'(t) = V_0 \sin 35° - 32t = 0$

when $t = \dfrac{V_0 \sin 35°}{32} = \dfrac{54.1 \sin 35°}{32} = .97$ sec.

At this time; $y(.97) = 54.1 \sin 35° (.97) - 16(.97)^2 + 7$

$\doteq 22$ ft

(c) $t = \dfrac{90}{V_0 \cos 35°} = \dfrac{90}{54.1 \cos 35°} \doteq 2$ sec.

$$\text{speed}^2 = b^2\{w^2(1-\cos wt)^2 + w^2 \sin^2 wt\}$$

$$= b^2w^2\{1 - 2\cos wt + \cos^2 wt + \sin^2 wt\} =$$

$$2b^2w^2\{1 - \cos wt\}$$

so $\text{speed} = \sqrt{2}\, bw \sqrt{1 - \cos wt}$

speed $= 0$ when $wt = 0, 2\pi, 4\pi, \ldots, 2\pi k \ldots$

i.e. when $t = \dfrac{2\pi k}{w}$, where k is any non-negative integer.

Max speed occurs when $1 - \cos wt$ is as large as possible

— i.e when $\cos wt = -1$, i.e when $wt = \pi, 3\pi, 5\pi, \ldots$

i.e. $t = \dfrac{(2k+1)\pi}{w}$, where $k = 0, 1, 2, \ldots$

Note: At these times the max speed $= \sqrt{2}\, bw \sqrt{2} = 2bw$

(b) 55 m.p.h linear speed $= 55(5280) = 290400$ ft/hr

$\longrightarrow \dfrac{290400}{2\pi}$ rev/hr since circumf of circle

$= 2\pi(1) = 2\pi$ ft. i.e. $w = 290400$ rad/hr.

by result of part (a), max speed $= 2(1)(290400)$ ft/hr

$= 110$ mi/hr, twice the linear speed of the car.

(6) By a theorem $\|\vec{p}\| = \text{Const} \Longrightarrow \vec{p} \cdot \vec{p}\,' = 0$ (this just says for any smooth circular motion, the radial vector and velocity (tangent) vector are always orthogonal). Now just apply this theorem with $\vec{p} = \vec{R}\,'$. So $\|\vec{R}\,'\| = \text{Const} \Longrightarrow$

$$\underbrace{\vec{R}\,'}_{\vec{v}} \cdot \underbrace{\vec{R}\,''}_{\vec{a}} = 0.$$

(7) Since the direction is always the same, say in the direction of a unit vector $\vec{u}$, $\vec{R}(t) = f(t)\,\vec{u}$, where $f(t) =$ distance at time $t \geq 0$. (so $f(t) \geq 0$)

Now speed $= \|\vec{R}\,'(t)\| = \|f'(t)\,\vec{u}\| =$ Constant.

So $|f'(t)|\underbrace{\|\vec{u}\|}_{1} = \overset{=\text{Const}}{f'(t)}$, since $f'(t) > 0$ (dist increasing)

So Acceleration $= \vec{a} = \dfrac{d}{dt}\left(f'(t)\vec{u}\right) = f''(t)\,\vec{u}$

$$= 0\,\vec{u} = \vec{0}$$