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Fantastic Functions of Several Variables

1. Describe the domain and range of each function of several variables:

   $(a)\ f(x,y) = \sqrt{4 - x^2 - y^2}$
   $(b)\ f(x,y) = Arc\sin(x + y)$
   $(c)\ f(x,y) = \ln(4 - x - y)$
   $(d)\ f(x,y) = e^{x/y}$
   $(e)\ f(x,y) = \dfrac{1}{xy}$

2. Sketch some of the level curves for each of the following functions of several variables that correspond to the given c-values:

   $(a)\ f(x,y) = e^{1 - x^2 - y^2}$, $c = -1,0,2,4$
   $(b)\ f(x,y) = \ln|y - x^2|$, $c = -3,0,5,10$
   $(c)\ z = x + y$, $c = -1,0,2,4$
   $(d)\ z = \sqrt{25 - x^2 - y^2}$, $c = 0, 1, 2, 3,$
   $(e)\ g(x,y) = \dfrac{x}{x^2 + y^2}$, $c = \pm 1/2, \pm 1, \pm 2$

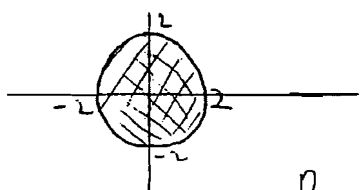
Solutions for Fantastic Functions of Several Variables

① (a) $f(x,y) = \sqrt{4-x^2-y^2}$

$4 - x^2 - y^2 \geq 0$

$4 \geq x^2+y^2$, so $x^2+y^2 \leq 4$

Domain $= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 4\}$ = closed disc of radius 2, centered at origin.



Range: $z = f(x,y) = \sqrt{4-(x^2+y^2)}$

Observe the smallest x^2+y^2 can be is 0, so the largest z can be is $\sqrt{4} = 2$. Also the smallest z can be is 0, which occurs when $x^2+y^2 = 4$. z is a continuous function of two variables, hence takes on all values in the interval $[0,2]$.

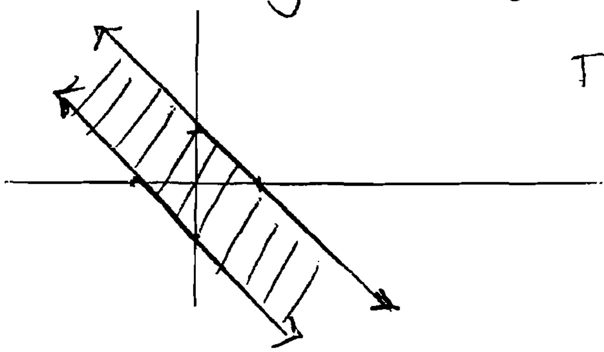
Note by squaring both sides one obtains $z^2 = 4 - x^2 - y^2$ i.e. $x^2+y^2+z^2 = 4$, a sphere of radius 2. Since $f(x,y)$ is defined by a positive square root, the 3-D graph of $f(x,y) = \sqrt{4-x^2-y^2}$ is only the upper hemisphere (including the boundary circle $x^2+y^2 = 4$).

(b) $f(x,y) = \text{Arcsin}(x+y)$

The domain of $y = \text{Arcsin}\, x$ is $[-1,1]$, so

$-1 \le x+y \le 1$. A graph of this in the xy-plane looks like:

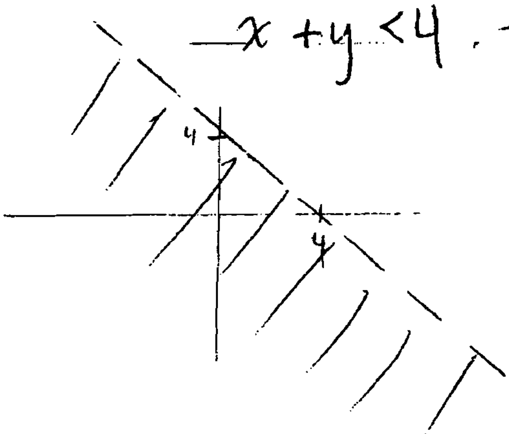
The range is $[-\frac{\pi}{2}, \pi/2]$.



(c) $f(x,y) = \ln(4-x-y)$. The domain of $y = \ln x$ is $x > 0$.

~~so~~ $4-x-y > 0$, so $4 > x+y$. i.e.,

$x+y<4$. The range is $(-\infty, \infty)$



(d) $f(x,y) = e^{x/y}$

Domain $= \{(x,y) : y \neq 0\}$ = Entire plane, except for x-Axis.

Range $= (0, \infty)$

(e) $f(x,y) = \dfrac{1}{xy}$

Domain $= \{(x,y) : x \neq 0 \text{ and } y \neq 0\}$ = "Entire plane, except for pts on the Coordinate Axes"

Range $= (-\infty, 0) \cup (0, \infty)$

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(2) (a) $f(x,y) = e^{1-x^2-y^2}$

For $c = -1$: $-1 = e^{1-x^2-y^2}$ never holds, so there is no level curve for $c = -1$. Similarly, $0 = e^{1-x^2-y^2}$ is never true, so there is no level curve for $c = 0$.

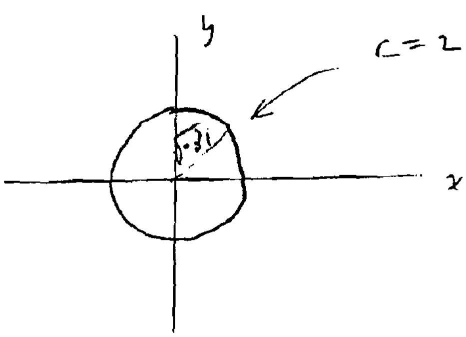
For $C = 2$; $2 = e^{1-x^2-y^2} \Longrightarrow \ln 2 = 1 - x^2 - y^2$

$\Longrightarrow .69 \doteq 1 - x^2 - y^2 \Longrightarrow x^2 + y^2 \doteq .31$

For $C = 4$: $4 = e^{1-x^2-y^2} \Rightarrow \ln 4 = 1 - x^2 - y^2$, so

$-.4 = x^2 + y^2$, which can never be true, since $x^2 + y^2 \geq 0$.

So there is no level curve for $c = 4$.



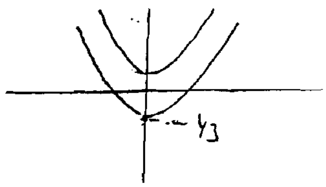
c=2 generates the only level curve for the specified c-values.

(b) $f(x,y) = \ln|y - x^2|$

$-3 = \ln|y-x^2| \Rightarrow e^{-3} = |y-x^2| \Rightarrow$

$e^{-3} = y - x^2$ or $-e^{-3} = y - x^2 \Rightarrow$

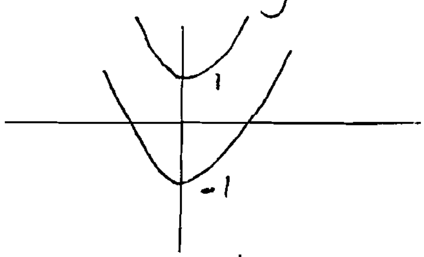
$y = x^2 + \frac{1}{e^3}$ or $y = x^2 - 1/e^3$



level curve for $C = -3$.

$0 = \ln|y-x^2| \Rightarrow 1 = |y-x^2| \Rightarrow y - x^2 = 1$ or $y - x^2 = -1$

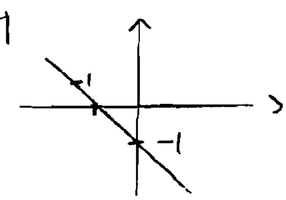
$\Rightarrow y = x^2+1$ or $y = x^2-1$



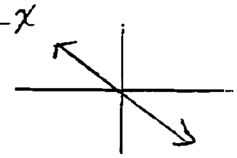
$\leftarrow$ level curve for $C = 0$

Similar level curves result when $c = 5$ and $c = 10$

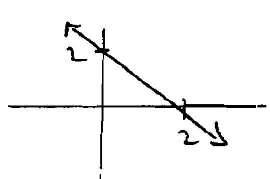
(c) $z = x + y$ For $c = -1$: $x + y = -1$



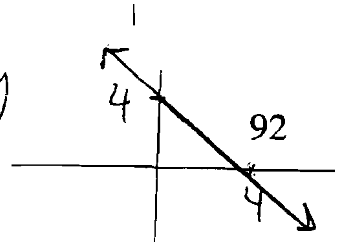
For $c = 0$: $0 = x + y \Rightarrow y = -x$



For $c = 2$: $2 = x + y$

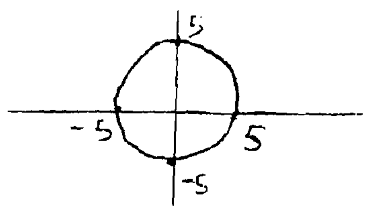


For $c = 4$: $4 = x + y$

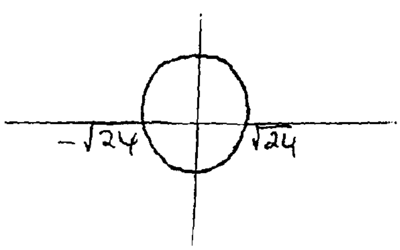


(d) $z = \sqrt{25 - x^2 - y^2}$

For $c = 0$: $0 = \sqrt{25-x^2-y^2} \implies 25 = x^2 + y^2$

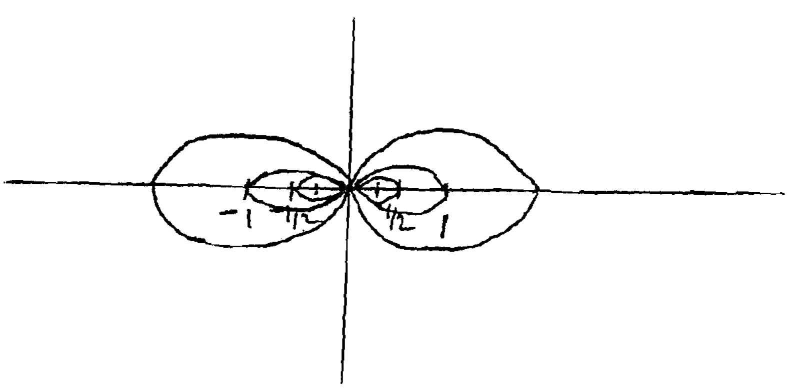


For $c = 1$: $1 = \sqrt{25 - x^2 - y^2} \implies 24 = x^2 + y^2$



$C = 2$ and $C = 3$ generate similar level curves (circles)

(e) $g(x,y) = \dfrac{x}{x^2 + y^2}$



A few level curves shown on the same set of axes.