Question 1

(a) Limit exists, value is 0.
(b) Limit exists, value is 1.
(c) Limit does not exist.
(d) Yes.
(e) No (it has a sharp point).

Question 2

(a) $\lim_{x\to 3} 4x^2 - 3\sin(\pi x) = 4(3)^2 - 3\sin(3\pi) = 36$.
(b) $\lim_{t\to 0} \frac{t}{\sin t} = \frac{1}{\lim_{t\to 0} \frac{\sin t}{t}} = \frac{1}{1} = 1$.
(c)

$$\lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{2-(2+h)}{h(2+h)2} = \lim_{h\to 0} \frac{-1}{2(2+h)} = -\frac{1}{4}.$$

Question 3

(a) $48x^{3-9x^2+14x-3}$.
(b) $2\cos(2x)$.
(c) $\tan x \sec x$ or $\frac{\sin x}{\cos^2 x}$.

Question 4

Using implicit differentiation,

$$2xy' + 2y - 8x + 6y^2y' = 0$$

so solving for y', we get

$$y' = \frac{8x - 2y}{2x + 6y^2}.$$

So at the point $(1, 1)$, the derivative (and hence the slope of the tangent to the curve) is

$$\frac{8-2}{2+6} = \frac{3}{4}$$

Question 5

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Question 6

(a) $5f'(1) + g'(1) = -5/3 - 8/3 = -13/3$.
(b) $f(0)g'(0) + g(0)f'(0) = 1/3 + 5 = 16/3$.
(c) $f'(g(0))g'(0) = f'(1)g'(0) = -1/9$.

Question 7

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h} \frac{\sqrt{2+h} + \sqrt{2}}{\sqrt{2+h} + \sqrt{2}} \end{aligned}$$

$$= \lim_{h\to 0} \frac{2+h-2}{h(\sqrt{2+h}+\sqrt{2})}$$
$$= \lim_{h\to 0} \frac{1}{(\sqrt{2+h}+\sqrt{2})}$$
$$= \frac{1}{2\sqrt{2}}$$

You could also do this by finding $f'(x)$ using limit techniques, and substituting $x = 1$ at the end, but that is more work.

Using the rules of differentiation,

$$f'(x) = \frac{d}{dx}(x+1)^{1/2} = \frac{1}{2}(x+1)^{-1/2}\frac{d}{dx}x = \frac{1}{2\sqrt{x+1}}$$

so

$$f'(1) = \frac{1}{2\sqrt{2}}$$

which agrees with the first part.

When $x = 1$, $y = f(1) = \sqrt{2}$, so the equation of the tangent line is

$$y - \sqrt{2} = \frac{1}{2\sqrt{2}}(x-1).$$

The slope of the normal is $-2\sqrt{2}$, so the equation of the normal is

$$y - \sqrt{2} = -2\sqrt{2}(x-1).$$

Question 8

Attacking the problem in a straightforward way,

$$h'(x) = \frac{1}{2}\left(\frac{(x-1)^2}{x+1}\right)^{-1/2} \frac{d}{dx}\frac{(x-1)^2}{x+1}$$
$$= \frac{1}{2}\left(\frac{(x-1)^2}{x+1}\right)^{-1/2} \frac{(x+1)\frac{d}{dx}(x-1)^2 - (x-1)^2\frac{d}{dx}(x+1)}{(x-1)^2}$$
$$= \frac{1}{2}\left(\frac{(x-1)^2}{x+1}\right)^{-1/2} \frac{(x+1)2(x-1) - (x-1)^2}{(x-1)^2}$$

Which would be sufficient to get full marks (it can be simplified a bit beyond this, but that is not important for the answer).

You can save yourself a little bit of work, by noticing that you can write

$$h(x) = ((x-1)^2(x+1)^{-1})^{1/2} = (x-1)(x+1)^{-1/2}$$

and then using the product rule rather than the quotient rule.

Question 9

See the text, Section 2.2, Proof of Rule 4, p123, but with f and g in the place of u and v, respectively.

Or see the notes you took in class on the rules of differentiation.