Final Exam

Math 151, Sections 501–503

May 11th, 1998

Version A

Name:

Section:

Student Number:

You are not permitted to use calculators on this exam.

Part I:
Part II:
Total:

Part I

Each problem is worth 4 points. Show all working. Write your final answer in the box provided.

1. Find the derivative of

$$f(x) = 3x^5 + 7x^3 + 2x^2 + 5x + 10.$$

2. Find

$$\int_0^1 x^3 + 2x^2 + 2\ dx.$$

3. Find

$$\lim_{x \to 1} \frac{x^2 + 2x}{3x^3 + 6}.$$

4. Find the tangent line at $x = 0$ of

$$f(x) = \frac{2x}{e^x + e^{-x}}.$$

5. Find the derivative of

$$h(t) = \int_{t^2}^{4} \cos(x)e^{-x^2}\ dx.$$

6. Find the unit vector tangent to the curve

$$\mathbf{r}(t) = \langle 2\cos 2t, 2\sin 2t\rangle$$

when $t = \pi/8$.

7. Find

$$\lim_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

(Hint: L'Hopital's rule won't work)

8. Find

$$\lim_{x\to 0^+} \ln x \sin x.$$

9. Find

$$\lim_{x \to 0} x^2 \cos(1/x).$$

10. Find

$$\int_0^1 \frac{5x}{\sqrt{1+3x^2}}\, dx.$$

11. Find

$$\int t^5 \sqrt{t^3+1}\, dt.$$

12. Find the inverse function of

$$f(x) = \frac{x-2}{x+2}.$$

13. If $\vec{a} = \langle 3, 4 \rangle$ and $\vec{b} = \langle -5, 12 \rangle$, find $\mathrm{proj}_{\vec{a}}\vec{b}$.

Part II

Each problem is worth 8 points.

1. If a body of mass m falls from rest at time 0, then the velocity at time $t \geq 0$ is given by

   $$v(t) = \frac{mg}{k}\left(1 - e^{-kt/m}\right)$$

   where k is the air resistance of the body and g is acceleration due to gravity (assume $g = 9.8 \text{ ms}^{-2}$). Bruce the Australian philosopher drops a gizmo with a mass of 3 kg and an air resistance of 0.5 kg s^{-1} off a tall cliff.

   (a) Find the formula for the acceleration of the gizmo at time t. (2 points)

   (b) Find the terminal velocity of the gizmo (ie. the limit of $v(t)$ as $t \to \infty$). (3 points)

   (c) Find the formula for the distance the gizmo has fallen after t seconds. (3 points)

2. A curve C is defined parametrically by

   $$x = t^2 \qquad y = t^3 - 3t$$

   (a) Determine on which intervals the curve rises and on which intervals it falls. (3 points)

   (b) Find where the tangent lines are horizontal and vertical. (3 points)

   (c) Sketch the curve. (2 points)

3. Consider the function $f(x) = \frac{x^3}{6} + \frac{x^2}{2} + x + 1$.

   (a) Show that f has at least one root. (3 points)

   (b) Show that f has at most one root. (5 points)

4. Two students, Bruce and Sheila, were asked to find $\frac{dy}{dx}$ for the curve $\frac{x^2}{y} + y = 3$.

   (a) Bruce worked the problem directly, and obtained the answer $\frac{2xy}{(x^2-y^2)}$. Check Bruce's working. (4 points)

   (b) Sheila realized that by multiplying both sides of the Cartesian equation by y to get $x^2 + y^2 = 3y$ made the differentiation easier. However the answer she got was $\frac{2x}{(3-2y)}$. Check her working. (2 points)

   (c) Did one of the students make a mistake, or are they both correct? If you think there was a mistake, explain what it was; if you think they are both correct, explain why the two answers are equal. (2 points)

5. A tow-rope is hitched to a car and run over a pulley 3m above a dock and then straight down to a crate sitting in a boat. If the car is travelling at 0.1 ms^{-1} when it is 4m from the pulley, how fast is the crate rising at that instant.

6. A piece of wire which is 10 cm long is cut into two pieces, one of which is bent into a square, the other into a circle. How should the wire be cut to (a) maximize, and (b) minimize, the total area of the shapes, and what is the maximum area?

Bonus Questions

These extra questions will be used to distinguish between students on the border-line between letter grades.