Question 1

There are a number of possible solutions to this question. The following is one, with the key points noted.

(Ooops, need to get the diagram into the computer somehow ... stay tuned ...)

If you had problems with this one, you need to review your intuitive understanding of limits, continuity and derivatives, particularly the "typical" pictures of when functions have or don't have or don't have limits, are or are not continuous, and are or are not differentiable

Question 2

(a) $\lim_{x\to 3} 5x^2 - 3\tan(\pi x) = 5 \times 3^2 - 3\tan(3\pi) = 45 - 0 = 45$ because everything is continuous.

(b) $\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} = 0$ (this is one of the standard trig limits).

(c) $\lim_{t\to 0} t\sin(1/t) = 0$, because $-|t| \le t\sin(1/t) \le |t|$, and $\lim |t| = 0 = \lim -|t|$, so the sandwich theorem tells us the limit is 0.

If you had difficulty with these you need to review how we calculate limits (Sections 1.2, 1.5 and 2.4).

Question 3

(a) $64x^3 + 24x + 3x^{-2}$

(b) The easy way:

$$\tan\theta\csc\theta = \frac{\sin\theta}{\cos\theta}\frac{1}{\sin\theta} = \frac{1}{\cos\theta} = \sec\theta,$$

so that

$$\frac{d}{d\theta}\tan\theta\csc\theta = \frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta$$

Other correct answers were given full credit.

(c) The easy way:

$$(2t^3 + 1)(7 - 3t) = -6t^4 + 14t^3 - 3t + 7$$

so that the derivative is:

$$-24t^3 + 42t^2 - 3$$

Other correct answers were given full credit.

If you had difficulty with these, you need to review how we calculate derivatives (Sections 2.2, 2.4).

Question 4

(a) When $x > 1$, $x - 1 > 0$, so $|x-1| = x - 1$. Therefore

$$\lim_{x \to 1+} \frac{|x-1|}{x-1} = \lim_{x \to 1+} \frac{x-1}{x-1} = 1$$

(b) When $x < 1$, $x - 1 < 0$, so $|x-1| = -(x-1)$. Therefore

$$\lim_{x \to 1-} \frac{|x-1|}{x-1} = \lim_{x \to 1+} \frac{-(x-1)}{x-1} = -1$$

(c) The two sided limit does not exist, since the left-hand and right-hand limits are different.
If you had difficulty with these, you need to review how left- and right-hand limits work (Section 1.4).

Question 5

$$h'(5) = \frac{g(5)f'(5) - f(5)g'(5)}{g(5)^2} = \frac{-3-5}{3^2} = -8/9$$

This question used the quotient rule in a fairly straightforward way (Section 2.2).

Question 6

C, since $f(1) = 2 > 0$ and $f(2) = -2 < 0$, the Intermediate Value Theorem tells us there must be a root in the interval $[1, 2]$.

This was an application of the IVT discussed at the end of section on continuous functions (Section 1.5).

Question 7

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h \to 0} \frac{\sqrt{1-(x+h)^2} - \sqrt{1-x^2}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{1-(x+h)^2} - \sqrt{1-x^2}}{h} \frac{\sqrt{1-(x+h_)^2} + \sqrt{1-x^2}}{\sqrt{1-(x+h)^2} + \sqrt{1-x^2}} \\
&= \lim_{h \to 0} \frac{1 - x^2 - 2xh - h^2 - 1 + x^2}{h(\sqrt{1-(x+h)^2} + \sqrt{1-x^2})} \\
&= \lim_{h \to 0} \frac{-2xh - h^2}{h(\sqrt{1-(x+h)^2} + \sqrt{1-x^2})} \\
&= \lim_{h \to 0} \frac{-2x - h}{(\sqrt{1-(x+h)^2} + \sqrt{1-x^2}} \\
&= \frac{-2x}{2\sqrt{1-x^2}} \\
&= \frac{-x}{\sqrt{1-x^2}}
\end{aligned}$$

This uses the limit definition of the derivative together with some tricks of calculating derivatives (Sections 1.2 and 2.1).

When $x = c$, $y = \sqrt{1-c^2}$, and $\frac{dy}{dx} = \frac{-c}{\sqrt{1-c^2}}$ the equation of the tangent line is

$$y - \sqrt{1-c^2} = \frac{-c}{\sqrt{1-c^2}}(x - c)$$

or (after a little algebra)

$$y = \frac{-c}{\sqrt{1-c^2}}x + \frac{1}{\sqrt{1-c^2}}$$

This uses the relationship between tangents and derivatives (Section 1.6 and 2.1).

If the line passes through $(\sqrt{2}, 0)$, then $x = \sqrt{2}$ and $y = 0$, so

$$0 = \frac{-\sqrt{2}c}{\sqrt{1-c^2}} + \frac{1}{\sqrt{1-c^2}}$$

and therefore

$$0 = -\sqrt{2}c + 1$$

and so $c = \frac{1}{\sqrt{2}}$. So the equation of the line passing through $(\sqrt{2}, 0)$ is

$$y = -x + \sqrt{2}.$$

If you got what the question was asking, this bit is just precalculus (Section P2 reviews the material). Also have a look at Section 2.1, question 52 for a similar problem from the homework.

Question 8

Attacking the problem in a straightforward way,

$$\begin{aligned} v(t) = s'(t) &= \frac{d}{dt}\left(\frac{4t}{t^2+1}\right) \\ &= \frac{(t^2+1)\frac{d}{dt}(4t) - (4t)\frac{d}{dt}(t^2+1)}{(t^2+1)^2} \\ &= \frac{(t^2+1)(4) - (4t)(2t)}{(t^2+1)^2} \\ &= \frac{4t^2+4-8t^2}{(t^2+1)^2} \\ &= \frac{4(1-t^2)}{t^2+1)^2} \end{aligned}$$

This is just the rules of differentiation—in particular the quotient rule—plus a little algebra (Section 2.2). Using limits works, but is overly complicated.

$$v(0) = \frac{4(1-0^2)}{(0^2+1)^2} = 4$$

and

$$v(1) = \frac{4(1-1^2)}{(1^2+1)^2} = 0$$

Just plug in ...

Question 9

Taking derivatives, we have

$$\begin{aligned}
\frac{d^2}{dx^2} &= \frac{d}{dx}\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) \\
&= \frac{d}{dx}\left(u\frac{dv}{dx}\right) + \frac{d}{dx}\left(v\frac{du}{dx}\right) \\
&= u\frac{d}{dx}\left(\frac{dv}{dx}\right) + \frac{dv}{dx}\frac{d}{dx}(u) + v\frac{d}{dx}\left(\frac{du}{dx}\right) + \frac{du}{dx}\frac{d}{dx}(v) \\
&= u\frac{d^2v}{dx^2} + \frac{dv}{dx}\frac{du}{dx} + \frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2} \\
&= u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}
\end{aligned}$$

This is just a matter of knowing how to apply the product rule to a term like $u\frac{dv}{dx}$ (Section 2.2).

We have $u = x^2 + 1$ and $v = \cos x$, so calculating derivatives:

$$\frac{du}{dx} = 2x$$

$$\frac{d^2u}{dx^2} = 2$$

$$\frac{dv}{dx} = -\sin x$$

$$\frac{d^2v}{dx^2} = -\cos x$$

Now we just plug in:

$$\frac{d^2y}{dx^2} = (x^2+1)(-\cos x) + 2(2x)(-\sin x) + (\cos x)2 = \cos x - x^2\cos x - 4x\sin x$$

Again, this is just knowing the rules of differentiation (Section 2.2).

Question 10

No working here (if you can do Q9, it's easy), but

$$\frac{d^3y}{dx^3} = u\frac{d^3v}{dx^3} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + \frac{d^3u}{dx^3}v$$

and

$$\frac{d^4y}{dx^4} = u\frac{d^4v}{dx^4} + 4\frac{du}{dx}\frac{d^3v}{dx^3} + 6\frac{d^2u}{dx^2}\frac{d^2v}{dx^2} + 4\frac{d^3u}{dx^3}\frac{dv}{dx} + \frac{d^4u}{dx^4}v.$$

The pattern of coefficients is:
1st derivative: 1 1
2nd derivative: 1 2 1
3rd derivative: 1 3 3 1
4th derivative: 1 4 6 4 1
which is Pascal's triangle.

This means that the general formula is (in summation notation):

$$\frac{d^ny}{dx^n} = \sum_{k=1}^{n} \frac{n!}{k!(n-k)!}\frac{d^ku}{dx^k}\frac{d^{n-k}v}{dx^{n-k}}$$