

3.3 The Graph of a Rational Function

This is an extension of the previous section. We will find the intercepts, asymptotes, and finally the graph.

EXAMPLE: Find the intercepts, asymptotes, and graph of $y = \frac{x+1}{x^2-9}$.

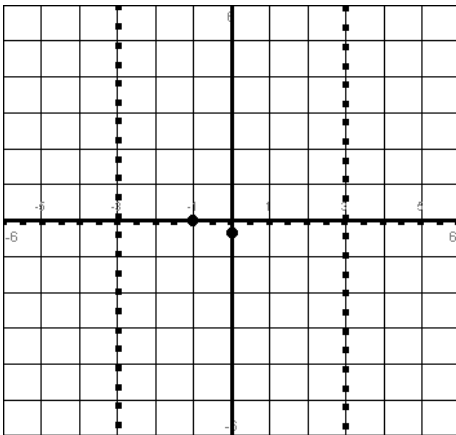
First we will find the x-intercept by setting the top equal to zero: $x + 1 = 0$ so $x = -1$. We write $(-1, 0)$.

To find the y-intercept, put in a zero for x: $y = \frac{0+1}{0^2-9}$. You will get $y = -\frac{1}{9}$ and we write $(0, -\frac{1}{9})$.

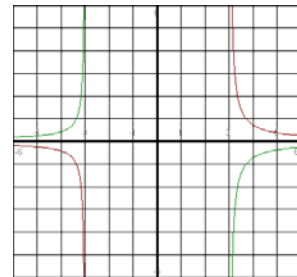
To find the vertical asymptote, set the bottom equal to zero: $x^2 - 9 = 0$ We get $x = \pm 3$.

For the horizontal asymptote, we notice the highest power on the top is less than the highest power on the bottom. From the previous section we know that the horizontal asymptote is automatically $y = 0$.

Now we are ready for the graph. First plot the intercepts. Then use dotted lines to draw in the asymptotes. The y-axis is one asymptote, so we have a horizontal dotted line at $y = 0$. The other asymptotes are $x = \pm 3$. These will be vertical lines going through 3 and -3 on the x-axis.



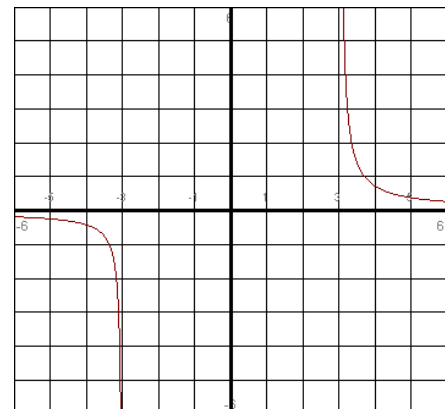
Now we need to graph. On the left and right sides of the graph we have two possible ways the graph can be drawn as the picture below shows. Notice that on the ends the graph will follow one asymptote, turn and follow the other asymptote. This is always the case with rational expressions:



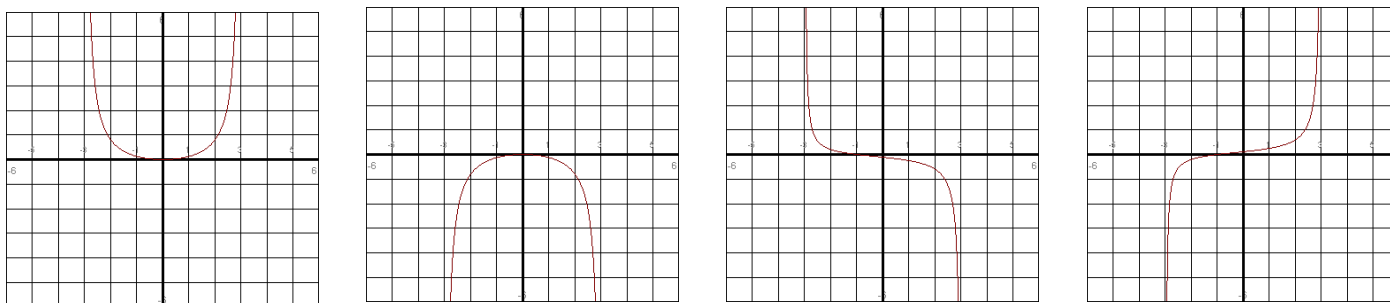
We need to choose whether the graph is above or below the x-axis on each end. In order to do this we need to choose a test points. We need to choose a point that is less than negative 3 since we want to know what the graph is doing to the left of vertical asymptote, $x = -3$. I will choose -4. We will put our test point into the original equation: $y = \frac{-4+1}{(-4)^2-9}$. We get $y = -\frac{3}{7}$. Since this number is negative I know that the graph must

be below the x-axis. Now let's test a point to the right of the vertical asymptote, $x = 3$. I will choose 4 since this is greater than 3. Once again we will put 4 in for x in the original equation: $y = \frac{4+1}{(4)^2-9}$

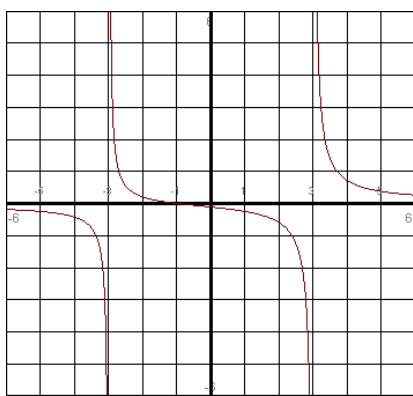
We get $y = \frac{5}{7}$ which is positive so this tells me the graph is above the x-axis. So now we know the graph looks like the following:



Now we need to take care of the middle part of the graph. In between the two vertical asymptotes a rational graph will **always** look like one of the following:



If we look at where our intercepts are that will help us choose one of the following. The only graph above that would fit the intercepts we are already plotted would be the third graph, so the following will be the our completed graph:



EXAMPLE: Find the intercepts, asymptotes, and graph of $y = \frac{2x^2 + 4x}{x^2 + 2x - 15}$.

Let's factor this first: $y = \frac{2x(x+2)}{(x-3)(x+5)}$. We should always factor something like this to see if anything cancels. Nothing does on this one so we will proceed to find the information.

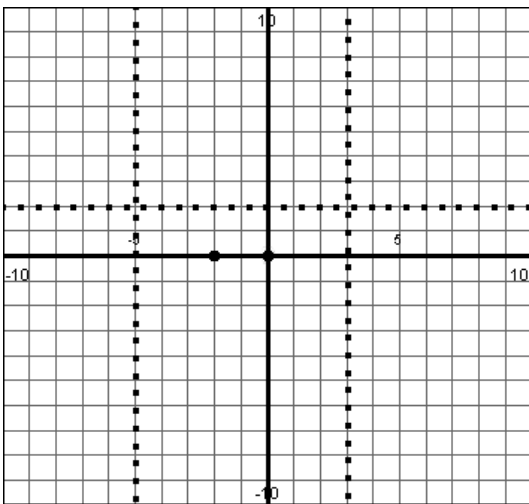
First we will find the x-intercept by setting the top equal to zero: $2x(x+2) = 0$. We will get $x = 0$ and $x = -2$. We will write our answer as $(0, 0)$ and $(-2, 0)$.

Since $(0, 0)$ is an x-intercept then automatically this is our y-intercept as well.

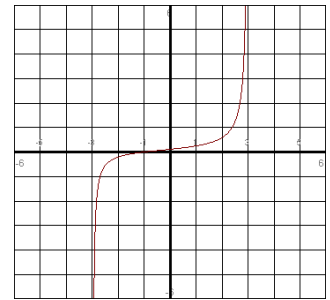
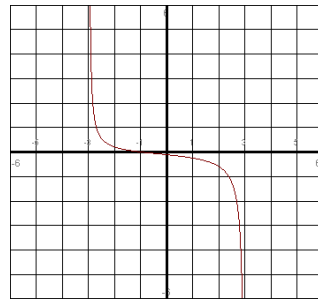
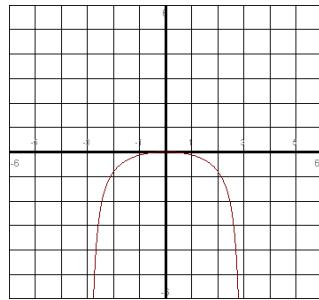
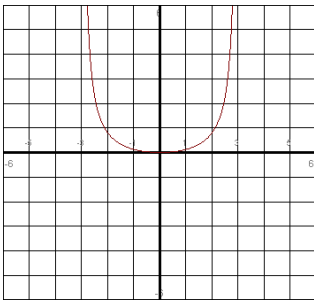
To find the vertical asymptote, set the bottom equal to zero: $(x-3)(x+5) = 0$. We get $x = 3$ and $x = -5$.

For the horizontal asymptote, notice that the highest power on top is the same as the highest power on the bottom, so we know that $y = \frac{a_n}{b_m}$. Here the $a_n = 2$ and $b_m = 1$, so the horizontal asymptote is $y = \frac{2}{1} = 2$.

Now we are ready for the graph. First plot the intercepts. Then use dotted lines to draw in the asymptotes. There will be a horizontal line through $y = 2$. The other asymptotes are $x = 3$ and $x = -5$. These will be vertical lines going through 3 and 5 on the x-axis.



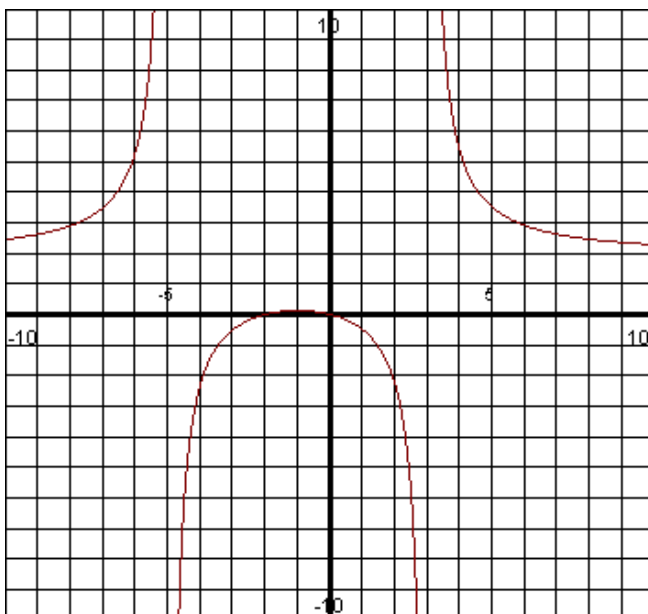
Now we are ready to draw the graph. Let's look at the part of the graph to the left of $x = -5$. We need to decide whether the graph will be above or below the horizontal asymptote. We actually do not need to use test points for this. Notice that our only intercepts are between the two vertical asymptotes. This means this is the only place the graph crosses the x-axis. So when we look at the part of the graph where x is less than -5 and since there are no intercepts there I know the graph has to be above the horizontal asymptote. The same reasoning can be said about the part of the graph where x is greater than 3 . There are no x-intercepts in the part of the graph where x is greater than 3 , so I know this part of the graph will also be above the horizontal asymptote. Now let's look between the two vertical asymptotes. Remember there are only four types of graphs that could appear here:



By the way the intercepts are located this tells me that only the first two graphs would work. How can we tell which one it is? We need to pick a test point. It can be any number between -5 and 3 . I will test $x = -3$. I will

put this in for x in the original equation: $y = \frac{2(-3)^2 + 4(-3)}{(-3)^2 + 2(-3) - 15} = \frac{7}{-30}$. Since this number is negative that

means when x is -3 the graph should be below the x-axis. This is only going to happen with the second graph above (upside down parabola). So now we can finish our graph:



EXAMPLE: Find the intercepts, asymptotes, and graph of $y = \frac{x}{x^2 - 4}$.

Let's factor this first: $y = \frac{x}{(x-2)(x+2)}$. Now we will find the x-intercept by setting the top equal to zero:

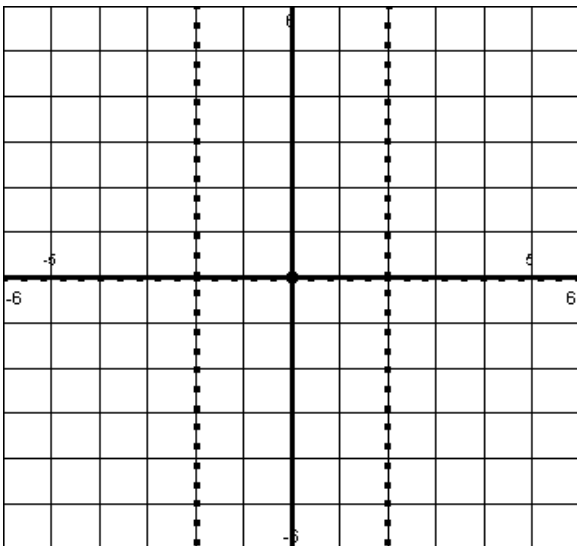
$x = 0$. We are done. We will write our answer as $(0, 0)$.

Again since $(0, 0)$ is an x-intercept then automatically this is our y-intercept as well.

To find the vertical asymptote, set the bottom equal to zero: $(x-2)(x+2) = 0$ We get $x = \pm 2$.

For the horizontal asymptote, notice that the highest power on top is less than the highest power on the bottom, so we know that the horizontal asymptote is automatically $y = 0$.

Now we are ready for the graph. We only have one point to plot this time. Then we have our dotted lines to draw in the asymptotes. There will be a horizontal line through $y = 0$. The other asymptotes are $x = \pm 2$. These will be vertical lines going through -2 and 2 on the x-axis.



Now we need draw the graph. Right now not much information is given. We need to use test an x value that is less than -2 and greater than -2 to see if the graph is above or below the x-axis. First I will test $x = -3$. We will put this in for x in the original equation: $y = \frac{-3}{(-3)^2 - 4} = \frac{-3}{5}$. We get a

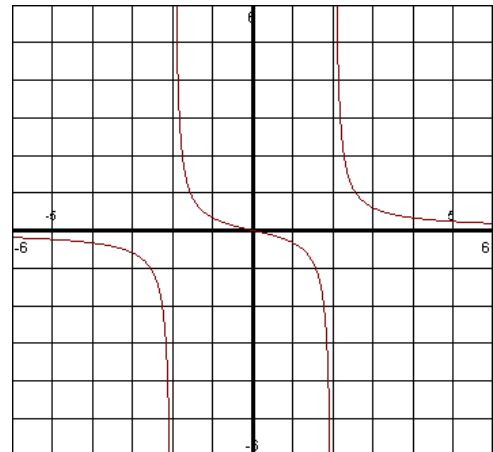
negative number so I know the graph is below the x-axis at this point. I also need to test a point greater than 2. I will choose $x = 3$. You will get $y = \frac{3}{3^2 - 4} = \frac{3}{5}$. So I know that

the graph must be above the x-axis. What about the part of the graph between the two vertical asymptotes? We need to do test points here as well to determine which of the four models the graph resembles. I will test $x = 1$ and $x = -1$.

We will not test those values: $y = \frac{1}{1^2 - 4} = -\frac{1}{3}$ This tells us that when $x = 1$ the graph is below the x-axis.

Now if we test $x = -1$ you will get $y = \frac{-1}{(-1)^2 - 4} = \frac{1}{3}$. So we know the graph is above the x-axis at $x = -1$.

This tells us that the only graph that will work in the middle is the third one.



EXAMPLE: Find the intercepts, asymptotes, and graph of $y = \frac{x^2 - 4}{x}$.

Let's factor this first: $y = \frac{(x-2)(x+2)}{x}$. Now we will find the x-intercept by setting the top equal to zero:

$$(x-2)(x+2) = 0. \text{ We get } x = \pm 2. \text{ We can write this as } (\pm 2, 0).$$

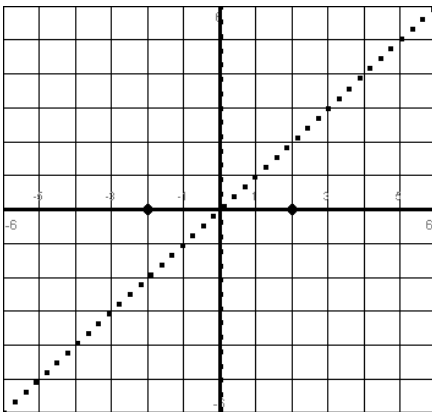
For the y-intercept you would put in a zero for x. If you do then you will be dividing by zero, so there is no y-intercept in this case.

To find the vertical asymptote, set the bottom equal to zero: $x = 0$. We only have one vertical asymptote here.

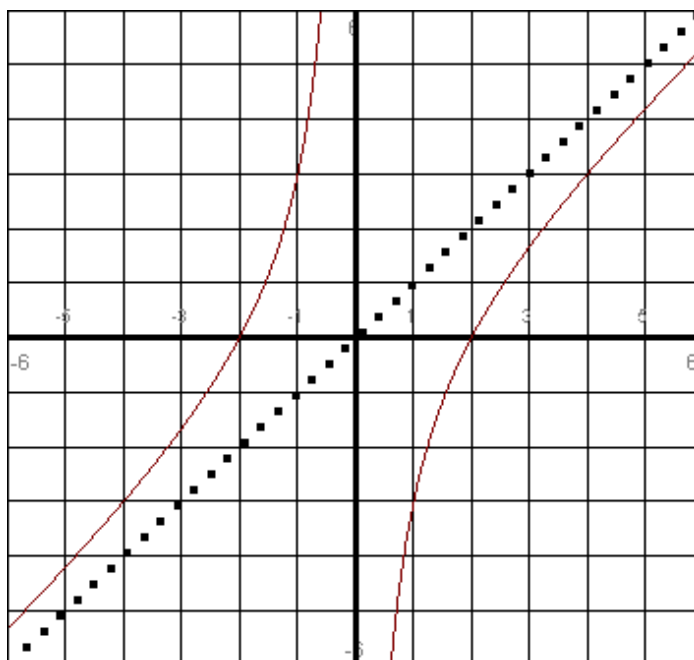
For the horizontal asymptote, notice that the highest power on top is more than the highest power on the bottom, so we know that this is no horizontal asymptote. However we need to find the oblique asymptote by doing long division:

$$\begin{array}{r} x+0 \\ x \overline{) x^2 + 0x - 4} \\ \underline{x^2} \\ 0x \\ \underline{0x} \\ -4 \end{array}$$

We need to put in a $0x$ term since it is missing. After we do the division we end up with an $x + 0$. This is the equation of the line that the graph will follow. We need to set up our graph by plotting the intercepts and the asymptotes. We will have one vertical asymptote at $x = 0$ and the oblique asymptote will be the line $y = x$.



We see where the graph will cross the x-axis. This tells us that the graph will be in these section. To the left of the vertical asymptote the graph will follow the line $y = x$ and turn towards $(-2, 0)$. Then it will follow the vertical asymptote. To the right of the vertical asymptote the graph will follow the vertical asymptote, turn through $(2, 0)$ and then follow the line $y = x$.



EXAMPLE: Find the intercepts, asymptotes, and graph of $y = \frac{3x^2 - 10x - 8}{x^2 - 16}$.

Let's factor this first: $y = \frac{(3x+2)(x-4)}{(x+4)(x-4)}$. Notice that we can cancel out the $x - 4$ from the top and bottom.

Whenever this happens you will have what is called a hole in the graph. Whatever part you can cancel you want to set this equal to zero. We will have $x - 4 = 0$. Solving it we get $x = 4$. This is not a vertical asymptote. There will be a hole in the graph at the x value of 4.

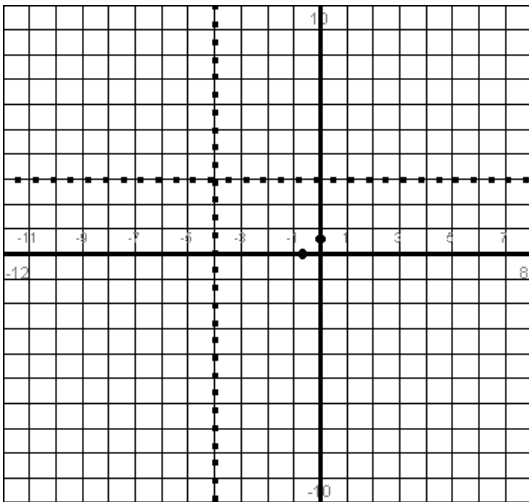
So the equation we will now look at to get the information will be $y = \frac{3x+2}{x+4}$.

To find the x -intercept by setting the top equal to zero: $3x+2=0$. We get $x = -\frac{2}{3}$, so we have $(-\frac{2}{3}, 0)$.

For the y -intercept you would put in a zero for x . If you do then you will have $y = \frac{3(0)+2}{(0)+4} = \frac{1}{2}$. So $(0, \frac{1}{2})$.

To find the vertical asymptote, set the bottom equal to zero: $x+4=0$ and we get $x = -4$. We only have one vertical asymptote here since the other term canceled out. We already determined that there will be a hole at the x value of 4.

To find the horizontal asymptote we see the highest power on top is the same as the highest power on the bottom. So the horizontal asymptote is $y = \frac{a_n}{b_m}$. Here the $a_n = 3$ and $b_m = 1$, so $y = \frac{3}{1} = 3$.



So we have drawn in the asymptotes and the intercepts. We notice that there are no x -intercepts to the right of $x = -4$. This tells us that the graph must be above the horizontal asymptote ($y = 3$). If the graph was below this line then it would have crossed the x -axis but we know this does not happen. To the right of the $x = -4$ we notice the graph does cross the x -axis so we know the graph is in this lower portion and not above $y = 3$. Of course if you are unsure where the graph should be you can always use test points. You test a point less than $x = -4$ and greater than $x = -4$. When we draw the graph we need to make sure that we put a hole when $x = 4$.

