3.5A Limits at Infinity

We are going to come back to limits. This time we will look at the limit of something as x goes to infinity. What you are really doing is finding the horizontal asymptote.

Definition of a Horizontal Asymptote:

The line $y = L$ is a horizontal asymptote of the graph of f if:

$\lim_{x\to-\infty} f(x) = L$ or $\lim_{x\to\infty} f(x) = L$

Limits at Infinity

$\lim_{x\to\infty} \frac{c}{x^r} = 0$ Where c is any real number and r is a positive rational number.

$\lim_{x\to-\infty} \frac{c}{x^r} = 0$

EXAMPLE: Find the limit: $\lim_{x\to-\infty} -4 + \frac{3}{x^2}$

We can break up this limit: $\lim_{x\to-\infty} -4 + \lim_{x\to-\infty} \frac{3}{x^2}$. This equals $-4 + 0 = -4$.

EXAMPLE: Find the limit: $\lim_{x\to\infty} \frac{3x+2}{x-4}$

To solve this we will divide each term in the numerator and denominator by the highest power of x we see in the <u>DENOMINATOR</u>. In this problem the highest power of x in the denominator will be x.

$\lim_{x\to\infty} \frac{\frac{3x}{x} + \frac{2}{x}}{\frac{x}{x} - \frac{4}{x}}$ Now we simplify.

$\lim_{x\to\infty} \frac{3 + \frac{2}{x}}{1 - \frac{4}{x}}$ Now we take the limit of each term separately:

$\frac{\lim_{x\to\infty} 3 + \lim_{x\to\infty} \frac{2}{x}}{\lim_{x\to\infty} 1 - \lim_{x\to\infty} \frac{4}{x}} = \frac{3+0}{1-0} = 3$ So $\lim_{x\to\infty} \frac{3x+2}{x-4} = 3$.

EXAMPLE: Find the limit: $\lim_{x\to\infty} \dfrac{3-2x}{3x^3-1}$

The highest power of x in the denominator is x^3, so we divide everything in the top and bottom by x^3.

$$\lim_{x\to\infty} \frac{\frac{3}{x^3}-\frac{2x}{x^3}}{\frac{3x^3}{x^3}-\frac{1}{x^3}}$$ Now simplify.

$$\lim_{x\to\infty} \frac{\frac{3}{x^3}-\frac{2}{x^2}}{3-\frac{1}{x^3}}$$ We will take the limit of each term individually.

$$\frac{\lim_{x\to\infty}\frac{3}{x^3}-\lim_{x\to\infty}\frac{2}{x^2}}{\lim_{x\to\infty}3-\lim_{x\to\infty}\frac{1}{x^3}}=\frac{0-0}{3-0}=0$$ So $\lim_{x\to\infty} \dfrac{3-2x}{3x^3-1}=0$.

EXAMPLE: Find the limit: $\lim_{x\to\infty} \dfrac{3-2x^2}{3x-1}$

The highest power of x in the denominator is x , so we divide everything in the top and bottom by x.

$$\lim_{x\to\infty} \frac{\frac{3}{x}-\frac{2x^2}{x}}{\frac{3x}{x}-\frac{1}{x}}$$ Now simplify.

$$\lim_{x\to\infty} \frac{\frac{3}{x}-2x}{3-\frac{1}{x}}$$ We will take the limit of each term individually.

$$\frac{\lim_{x\to\infty}\frac{3}{x}-\lim_{x\to\infty}2x}{\lim_{x\to\infty}3-\lim_{x\to\infty}\frac{1}{x}}=\frac{0-\infty}{3-0}=\frac{-\infty}{3}=-\infty$$ Negative infinity divided by any number is still negative infinity.

So $\lim_{x\to\infty} \dfrac{3-2x^2}{3x-1}=-\infty$.

EXAMPLE: Find the limit: $\lim_{x\to\infty} \dfrac{5x^{\frac{3}{2}}}{4x^{\frac{3}{2}}+1}$

The highest power of x in the denominator is $x^{\frac{3}{2}}$, so we divide everything in the top and bottom by $x^{\frac{3}{2}}$.

$$\lim_{x\to\infty} \frac{\frac{5x^{\frac{3}{2}}}{x^{\frac{3}{2}}}}{\frac{4x^{\frac{3}{2}}}{x^{\frac{3}{2}}}+\frac{1}{x^{\frac{3}{2}}}}$$

Now simplify

$$\lim_{x\to\infty} \frac{5}{4+\frac{1}{x^{\frac{3}{2}}}}$$

We will take the limit of each term individually.

$$\frac{\lim_{x\to\infty} 5}{\lim_{x\to\infty} 4+\lim_{x\to\infty} \frac{1}{x^{\frac{3}{2}}}} = \frac{5}{4+0} = \frac{5}{4}.$$

So $\lim_{x\to\infty} \dfrac{5x^{\frac{3}{2}}}{4x^{\frac{3}{2}}+1} = \dfrac{5}{4}$.

EXAMPLE: Find the limit: $\lim_{x\to\infty} \dfrac{5x^{\frac{3}{2}}}{4\sqrt{x}+1}$

The highest power of x in the denominator is $x^{\frac{1}{2}}$, so we divide everything in the top and bottom by $x^{\frac{1}{2}}$.

$$\lim_{x\to\infty} \frac{\frac{5x^{\frac{3}{2}}}{x^{\frac{1}{2}}}}{\frac{4x^{\frac{1}{2}}}{x^{\frac{1}{2}}}+\frac{1}{x^{\frac{1}{2}}}}$$

Now simplify

$$\lim_{x\to\infty} \frac{5x}{4+\frac{1}{x^{\frac{1}{2}}}}$$

We will take the limit of each term individually.

$$\frac{\lim_{x\to\infty} 5x}{\lim_{x\to\infty} 4+\lim_{x\to\infty} \frac{1}{x^{\frac{1}{2}}}} = \frac{\infty}{4+0} = \frac{\infty}{4} = +\infty.$$

So $\lim_{x\to\infty} \dfrac{5x^{\frac{3}{2}}}{4\sqrt{x}+1} = +\infty$.

EXAMPLE: Find the limit: $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}}$

For this one, you may think the highest power of x in the denominator is x^2, however we really have $\sqrt{x^2}$ which is really just x. If we start with the identity x = x and we square both sides we will get $x^2 = x^2$. Now take the square root of both sides and we will get $x = \pm\sqrt{x^2}$. So here is the rule. If the limit is going to positive infinity, use the definition $x = \sqrt{x^2}$. If the limit is going to negative infinity, use $x = -\sqrt{x^2}$.

For this example, we will let $x = \sqrt{x^2}$. Since the top is just x we will divide the top by just x. On the bottom we have a square root, so divide this by $\sqrt{x^2}$.

$$\lim_{x\to\infty} \frac{\frac{x}{x}}{\frac{\sqrt{x^2+1}}{\sqrt{x^2}}}$$ Now simplify.

$$\lim_{x\to\infty} \frac{1}{\sqrt{\frac{x^2+1}{x^2}}}$$ We can make this a single square root on the bottom.

$$\lim_{x\to\infty} \frac{1}{\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}}$$ We can break up the fraction and then simplify.

$$\lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}}$$ Now take the limit of each term individually.

$$\frac{\lim_{x\to\infty} 1}{\sqrt{\lim_{x\to\infty} 1+\lim_{x\to\infty}\frac{1}{x^2}}} = \frac{1}{\sqrt{1+0}} = 1$$ So $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}} = 1$.

EXAMPLE: Find the limit: $\lim_{x\to-\infty} \dfrac{x}{\sqrt{x^2+1}}$

Again the highest power in the denominator is x, but this time we will let $x = -\sqrt{x^2}$. We want to divide everything on the top by x and everything on the bottom by $-\sqrt{x^2}$.

$$\lim_{x\to-\infty} \frac{\frac{x}{x}}{\frac{\sqrt{x^2+1}}{-\sqrt{x^2}}}$$ Now simplify.

$$\lim_{x\to-\infty} \frac{1}{-\sqrt{\frac{x^2+1}{x^2}}}$$ Make a single root in the denominator. Now break up the fraction and simplify.

$$\lim_{x\to-\infty} \frac{1}{-\sqrt{1+\frac{1}{x^2}}}$$ Take the limit of each term individually.

$$-\frac{\lim_{x\to-\infty} 1}{\sqrt{\lim_{x\to-\infty} 1+\lim_{x\to-\infty} \frac{1}{x^2}}} = \frac{1}{-\sqrt{1+0}} = -1$$ So $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = -1$.

EXAMPLE: Find the limit: $\lim_{x\to-\infty} \frac{-3x+1}{\sqrt{x^2+x}}$

Again the highest power in the denominator is x, but this time we will let $x = -\sqrt{x^2}$. We want to divide everything on the top by x and everything on the bottom by $-\sqrt{x^2}$.

$$\lim_{x\to-\infty} \frac{\frac{-3x}{x}+\frac{1}{x}}{\frac{\sqrt{x^2+x}}{-\sqrt{x^2}}}$$ We can simplify this.

$$\lim_{x\to-\infty} \frac{-3+\frac{1}{x}}{-\sqrt{\frac{x^2+x}{x^2}}}$$ We made a single root in the denominator. Now separate the fraction and finish.

$$\lim_{x\to-\infty} \frac{-3+\frac{1}{x}}{-\sqrt{1+\frac{1}{x}}} = \frac{\lim_{x\to-\infty} -3+\lim_{x\to-\infty} \frac{1}{x}}{-\sqrt{\lim_{x\to-\infty} 1+\lim_{x\to-\infty} \frac{1}{x}}} = \frac{-3+0}{-\sqrt{1+0}} = 3,$$ so $\lim_{x\to-\infty} \frac{-3x+1}{\sqrt{x^2+x}} = 3$

Special limits with sine and cosine

$\lim_{x\to\infty} \sin x = \infty$ $\quad\quad$ $\lim_{x\to\infty} \frac{\sin x}{x} = 0$ $\quad\quad$ $\lim_{x\to\infty} \frac{\cos x}{x} = 0$

EXAMPLE: Find the limit: $\lim_{x\to\infty} \frac{x-\cos x}{x}$

$\lim_{x\to\infty} \frac{x}{x} - \frac{\cos x}{x}$ First we will separate the fraction

$\lim_{x\to\infty} 1 - \frac{\cos x}{x}$ After simplifying, take the limit of each term separately.

$\lim_{x\to\infty} 1 - \lim_{x\to\infty} \frac{\cos x}{x}$ Calculate the limit

$1 - 0 = 1$ So $\lim_{x\to\infty} \frac{x-\cos x}{x} = 1$

EXAMPLE: Find the limit: $\lim_{x\to\infty} \cos\left(\frac{1}{x}\right)$

We know that $\lim_{x\to\infty} \frac{1}{x} = 0$, so we will have $\lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = \cos(0) = 1$.

EXAMPLE: Find the limit: $\lim_{x\to\infty} \frac{4}{3x - \sin x}$

First divide the top and bottom by x since that is the highest power in the denominator.

$\lim_{x\to\infty} \frac{\frac{4}{x}}{\frac{3x}{x} - \frac{\sin x}{x}}$ Now simplify and take the limit of each term individually.

$$\lim_{x\to\infty} \frac{\frac{4}{x}}{3 - \frac{\sin x}{x}} = \frac{\lim_{x\to\infty} \frac{4}{x}}{\lim_{x\to\infty} 3 - \lim_{x\to\infty} \frac{\sin x}{x}} = \frac{0}{3-0} = 0$$