3.6 Summary of Curve Sketching

Now we will put everything together we have learned in the previous sections and use this to sketch the graph.

EXAMPLE: Find all extrema, interval(s) of increasing/decreasing, critical points, interval(s) of concavity, inflection points, intercepts, asymptotes and use this information to graph $f(x) = -\frac{1}{3}x^3 + x - \frac{2}{3}$.

First we will find the first derivative to find any critical points: $f'(x) = -x^2 + 1$. Setting this equal to zero we get $x = \pm 1$. We can set up the table and use test points with the FIRST DERIVATIVE:

-	+	-

-1 1

We see that the interval of increasing is (-1, 1) and the intervals of decreasing are $(-\infty, -1) \cup (1, \infty)$. The relative minimum is at $\left(-1, -\frac{4}{3}\right)$ and the relative maximum is at (1, 0). I got the y values for these points by using the ORIGINAL function.

The second derivative is $f''(x) = -2x$. Setting this equal to zero we will get x = 0. We can put this on our table and use test points with the SECOND DERIVATIVE:

+	-

0

We see that the graph is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. The inflection point is at $\left(0, -\frac{2}{3}\right)$. This is also our y-intercept.

We found that the relative maximum was at (1, 0). There is an x-intercept, but we need to see if there are any other intercepts. So we need to solve $0 = -\frac{1}{3}x^3 + x - \frac{2}{3}$. Multiplying both sides by -3 we get: $0 = x^3 - 3x + 2$. Now we can use synthetic division for precalculus:

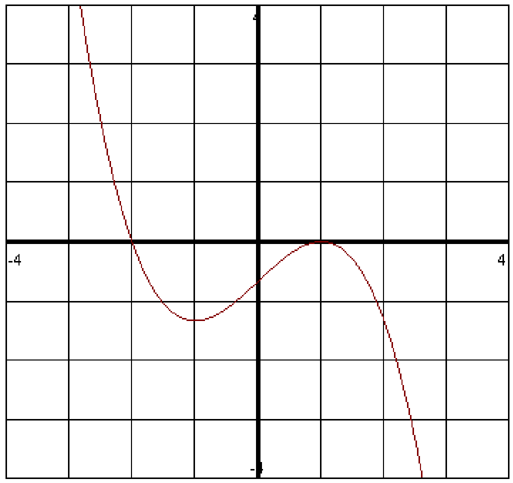
1 |	1	0	-3	2
		1	1	-2
	1	1	-2	0

So now we need to solve $x^2 + x - 2 = 0$. Factoring we get $(x-1)(x+2) = 0$, so x = 1 and x = -2. So now we found our second x-intercept which is at (-2, 0).

Let's now see what the graph does as x gets really big or really small (end behavior).

$$\lim_{x \to \infty}\left(-\frac{1}{3}x^3 + x - \frac{2}{3}\right) = -\infty \qquad \lim_{x \to -\infty}\left(-\frac{1}{3}x^3 + x - \frac{2}{3}\right) = \infty$$

Now we are ready to graph our function. Since the graph goes to positive infinity as x gets smaller I know the graph will be coming down from the upper left part of my graph. It will pass through the points (-2, 0), $\left(-1,\ -\frac{4}{3}\right)$, $\left(0,\ -\frac{2}{3}\right)$, and (1, 0). Then it will go down and to the right since the graph goes to negative infinity as x gets large. We also know the graph will not pass through at (1, 0) since this is a relative max.



EXAMPLE: Find all extrema, interval(s) of increasing/decreasing, critical points, interval(s) of concavity, inflection points, intercepts, asymptotes and use this information to graph $f(x) = x(x-2)^3$.

First we will find the first derivative to find any critical points by using the product rule: $f'(x) = x \cdot 3(x-2)^2(1) + (x-2)^3(1)$. We can factor out a common factor of $(x-2)^2$ to get $f'(x) = (x-2)^2[3x + (x-2)]$. Simplifying we will get $f'(x) = (x-2)^2(4x-2)$, or $f'(x) = 2(x-2)^2(2x-1)$. Setting this equal to zero we get $x = \frac{1}{2},\ 2$. We can set up the table and use test points with the FIRST DERIVATIVE:

-	+	+
	½	2

We see that the interval of increasing is $\left(\frac{1}{2},\ 2\right) \cup (2,\ \infty)$ and the intervals of decreasing are $\left(-\infty,\ \frac{1}{2}\right)$. The relative minimum is at $\left(\frac{1}{2},\ -\frac{27}{16}\right)$. There is no relative extrema at x = 2.

In order to get the second derivative you will need to use the product rule again. We will use $f'(x) = 2(x-2)^2(2x-1)$ and take its derivative: $f''(x) = 2(x-2)^2(2) + (2x-1) \cdot 4(x-2)(1)$. After factoring out a common factor of $4(x-2)$ you will get: $f''(x) = 4(x-2)[(x-2) + (2x-1)]$ which simplifies to $f''(x) = 4(x-2)(3x-3)$ or $f''(x) = 12(x-2)(x-1)$. Setting this equal to zero we will get x = 2. and x = 1. We can put this on our table and use test points with the SECOND DERIVATIVE:

+	-	+

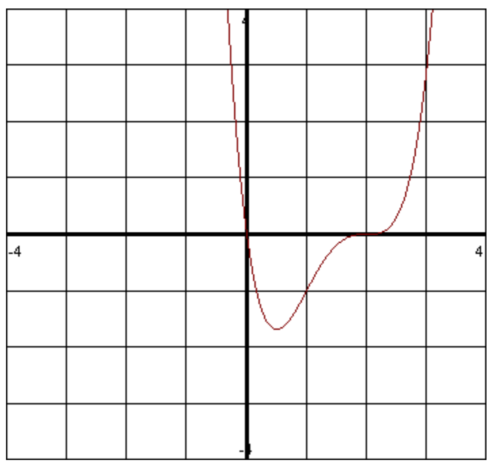
1 2

We see that the graph is concave down on (1, 2) and concave up on $(-\infty,\ 1)\cup(2,\ \infty)$. The inflection point is at $(1,\ -1)$ and (2, 0). We also know that (2, 0) is an x-intercept. The other one is at (0, 0). The y-intercept is also (0, 0).

Let's now see what the graph does as x gets really big or really small (end behavior).

$\lim_{x\to\infty}\left(x(x-2)^3\right)=\infty$ $\qquad$ $\lim_{x\to-\infty}\left(x(x-2)^3\right)=\infty$

Putting this all together we get the following graph. Notice that at (2, 0) there is a bend in the graph. This is not a relative max or min, however there is an inflection point here.



EXAMPLE: Find all extrema, interval(s) of increasing/decreasing, critical points, interval(s) of concavity, inflection points, intercepts, asymptotes and use this information to graph $f(x)=3x^4-6x^2+\frac{5}{3}$.

First we will find the first derivative: $f'(x)=12x^3-12x$. We can factor to get $f'(x)=12x(x+1)(x-1)$. Setting this equal to zero we get x = 0, 1, -1. We can set up the table and use test points with the FIRST DERIVATIVE:

-	+	-	+

-1 0 1

We see that the interval of increasing is $(-\infty,\ -1)\cup(0,\ 1)$ and the intervals of decreasing are $(-1,\ 0)\cup(1,\ \infty)$. The relative minimum is at $\left(-1,\ -\frac{4}{3}\right)$ and $\left(1,\ -\frac{4}{3}\right)$. The relative max is at $\left(0,\ \frac{5}{3}\right)$.

The second derivative is $f''(x)=36x^2-12$. Factoring will give us $f''(x)=12(3x^2-1)$. Setting this equal to zero we will get $x=\pm\frac{\sqrt{3}}{3}$. We can put this on our table and use test points with the SECOND DERIVATIVE:

+	-	+

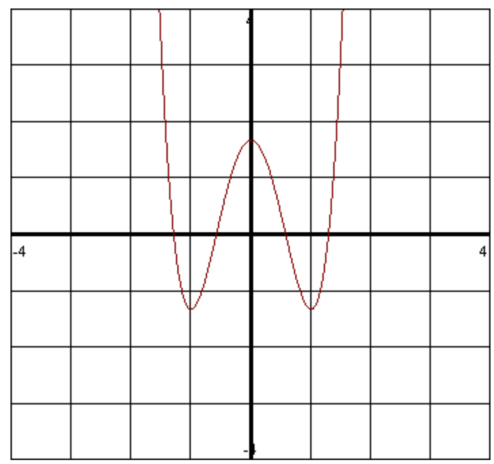
$-\frac{\sqrt{3}}{3}$ $\frac{\sqrt{3}}{3}$

We see that the graph is concave down on $\left(-\frac{\sqrt{3}}{3},\ \frac{\sqrt{3}}{3}\right)$ and concave up on $\left(-\infty,\ -\frac{\sqrt{3}}{3}\right)\cup\left(\frac{\sqrt{3}}{3},\ \infty\right)$. The inflection point is at $\left(-\frac{\sqrt{3}}{3},\ 0\right)$ and $\left(\frac{\sqrt{3}}{3},\ 0\right)$. These are also our x-intercepts. The y-intercept is at $\left(0,\ \frac{5}{3}\right)$.

Let's now see what the graph does as x gets really big or really small (end behavior).

$\lim_{x\to\infty}\left(3x^4-6x^2+\frac{5}{3}\right)=\infty$ $\lim_{x\to-\infty}\left(3x^4-6x^2+\frac{5}{3}\right)=\infty$

Putting this all together we get the following graph:



EXAMPLE: Find all extrema, interval(s) of increasing/decreasing, critical points, interval(s) of concavity, inflection points, intercepts, asymptotes and use this information to graph $f(x)=\frac{x}{x^2+1}$.

First we will find the first derivative using the quotient rule: $f'(x)=\frac{(x^2+1)(1)-x(2x)}{(x^2+1)^2}$. This simplifies to:

$f'(x)=\frac{1-x^2}{(x^2+1)^2}$. The derivative will not be undefined since the bottom can never be zero, so the only way to find the critical points is to set the derivative to zero and we get will get x = 1, -1. We can set up the table and use test points with the FIRST DERIVATIVE:

-	+	-

-1 1

We see that the interval of increasing is (-1, 1) and the intervals of decreasing are $(-\infty,\ -1)\cup(1,\ \infty)$. The relative minimum is at $\left(-1,\ -\frac{1}{2}\right)$. The relative max is at $\left(1,\ \frac{1}{2}\right)$.

The second derivative also involves the quotient rule: $f''(x) = \dfrac{(x^2+1)^2(-2x) - (1-x^2)\cdot 2(x^2+1)(2x)}{(x^2+1)^4}$. We can factor out a common factor of $-2x(x^2+1)$ from the numerator to get $f''(x) = \dfrac{-2x(x^2+1)[(x^2+1)+2(1-x^2)]}{(x^2+1)^4}$.

Simplifying the numerator you will get: $f''(x) = \dfrac{-2x(x^2+1)(3-x^2)}{(x^2+1)^4}$ Setting this equal to zero we will get $x = \pm\sqrt{3},\ 0$. We can put this on our table and use test points with the SECOND DERIVATIVE:

-	+	-	+

$-\sqrt{3}$ 0 $\sqrt{3}$

We see that the graph is concave down on $\left(-\infty,\ -\sqrt{3}\right)\cup\left(0,\ \sqrt{3}\right)$ and concave up on $\left(-\sqrt{3},\ 0\right)\cup\left(\sqrt{3},\ \infty\right)$. The inflection points are at $\left(-\sqrt{3},\ -\dfrac{\sqrt{3}}{4}\right)$, $(0,\ 0)$, $\left(\sqrt{3},\ \dfrac{\sqrt{3}}{4}\right)$.

Let's now see what the graph does as x gets really big or really small (end behavior).

$\lim\limits_{x\to\infty}\left(\dfrac{x}{x^2+1}\right) = 0$ $\lim\limits_{x\to-\infty}\left(\dfrac{x}{x^2+1}\right) = 0$

This means we have a horizontal asymptote at y = 0.

Putting this all together we get the following graph: