5.6 Inverse Trigonometric Functions: Differentiation

From trigonometry we know that $\sin 30^\circ = \frac{1}{2}$. We put in an angle and get a value as a result. In inverse trig functions we put in the value and get an angle: $\sin^{-1}\frac{1}{2} = 30^\circ$. So here we put in the value of one half and got 30 degrees as a result. We are not allowed to put any number into our inverse trig functions. There are restrictions on the domain that are given in the following table:

	Domain	Range
$y = \arcsin x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \arccos x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = arc\cot x$	$-\infty < x < \infty$	$0 \le y \le \pi$
$y = arc\sec x$	$\|x\| \ge 1$	$0 \le y \le \pi,\ \ y \ne \frac{\pi}{2}$
$y = ar\csc x$	$\|x\| \ge 1$	$-\frac{\pi}{2} < y < \frac{\pi}{2},\ \ y \ne 0$

We are not going to look at the graphs of the inverse trig functions in this class, however they are on pg 372 in case you want to see them.

EXAMPLE: Evaluate (if possible): $\arccos(0)$.

What this is asking us to do is to find the angle that gives us a cosine value of 0. Our answer needs to be between 0 and π since that is our range for an inverse cosine. You can look at the unit circle for this one. You will find that the answer is $\frac{\pi}{2}$.

EXAMPLE: Evaluate (if possible): $\arcsin\left(\frac{99}{98}\right)$.

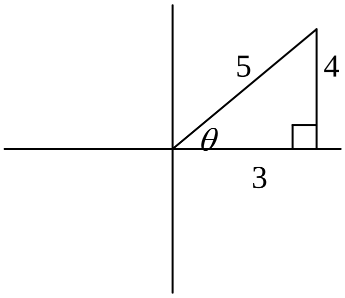
This is not possible (undefined) because this is outside the domain for the inverse sine, which is $-1 \le x \le 1$. The fraction $\frac{99}{98}$ is greater than 1.

EXAMPLE: Evaluate (if possible): $arc\cot(-\sqrt{3})$.

What this is asking us to do is to find the angle that gives us a cotangent value of 0. Our answer needs to be between 0 and π since that is our range for an inverse cosine. Since cotangent is cosine divided by sine, we can find a point on the unit circle such that the cosine over sine gives us a value of $-\sqrt{3}$. The answer is $\frac{5\pi}{6}$.

EXAMPLE: Find the exact value: $\cot\left(arc\csc\left(\frac{5}{4}\right)\right)$.

These problems involve drawing a triangle and labeling the sides. The inverse trig function will tell you where to draw the triangle. In our example there is an inverse cosecant. The inverse cosecant's range will tell us where we can draw the triangle. We know the range for the inverse cosecant is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. This corresponds to the first and fourth quadrant. Since the fraction $\frac{5}{4}$ is positive, the only quadrant the triangle can be drawn in is the first quadrant. We know that the hypotenuse is 5 and the opposite side is 4. The Pythagorean Theorem will give us the opposite side, which is 3.

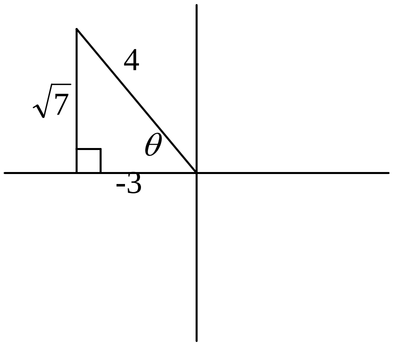


The cotangent on the outside of our problem tells us how to write our answer. From our drawing, cotangent is 3 over 4, so we write our answer as:

$$\cot\left(arc\csc\left(\frac{5}{4}\right)\right) = \frac{3}{4}$$

EXAMPLE: Find the exact value: $\sec\left(arc\cot\left(-\frac{3}{\sqrt{7}}\right)\right)$.

The inverse trig function will tell you where to draw the triangle, and in our case there is an inverse cotangent. The inverse cotangent's range will tell us where we can draw the triangle. We know the range for the inverse cotangent is $0 \le y \le \pi$. This corresponds to the first and second quadrant. Since the fraction inside the inverse is negative, the only quadrant the triangle can be drawn in is the second quadrant. We know that the adjacent side is -3 and the opposite side is $\sqrt{7}$. The Pythagorean Theorem will give us the hypotenuse which is 4.

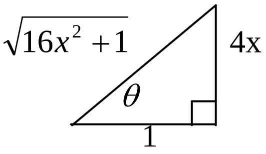


The secant on the outside of our problem tells us how to write our answer. From our drawing, tangent is $\sqrt{7}$ over -3, so:

$$\sec\left(arc\cot\left(-\frac{3}{\sqrt{7}}\right)\right) = -\frac{4}{3}$$

EXAMPLE: Write in algebraic form: $\sec(\arctan 4x)$.

These problems involve drawing a triangle and labeling the sides with algebraic expressions. For all these problems we will assume that x is positive and the triangle should be drawn in the first quadrant. We can rewrite our problem as: $\sec\left(\arctan\frac{4x}{1}\right)$ We know that the adjacent side is 1 and the opposite side is 4x. We can use the Pythagorean theorem to find the hypotenuse: $c^2 = (4x)^2 + (1)^2$. So we have $c = \sqrt{16x^2 + 1}$

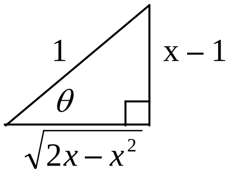


The secant on the outside of our problem tells us how to write our answer. From our drawing, secant is $\sqrt{16x^2+1}$ over 1 so we write our answer as:

$$\sec(\arctan 4x) = \sqrt{16x^2 + 1}$$

EXAMPLE: Write in algebraic form: $\cot(\arcsin(x-1))$.

These problems involve drawing a triangle and labeling the sides with algebraic expressions. For all these problems we will assume that x is positive and the triangle should be drawn in the first quadrant. We can rewrite our problem as: $\sec\left(\arcsin\frac{x-1}{1}\right)$ We know that the opposite side is x – 1 and the hypotenuse is 1. We can use the Pythagorean theorem to find the hypotenuse: $1^2 = (x-1)^2 + a^2$. So we have $1 = x^2 - 2x + 1 + a^2$. Solving for a^2 we get $a^2 = 2x - x^2$, so $a = \sqrt{2x - x^2}$. Now we can draw the triangle.

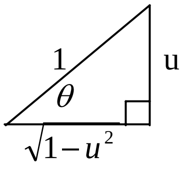


The cotangent on the outside of our problem tells us how to write our answer. From our drawing, cotangent is $\sqrt{2x-x^2}$ over x – 1 so we write our answer as:

$$\cot(\arcsin(x-1)) = \frac{\sqrt{2x-x^2}}{x-1} \text{ where } x \neq 1.$$

Derivative of inverse trig functions

Let's find the derivative of $\arcsin u$. In order to do this we will draw a triangle like in the previous problems. We can rewrite this problem as $\arcsin\frac{u}{1}$. Then we know the hypotenuse is 1 and the opposite side is u. Then we can use the Pythagorean Theorem to find the remaining side. You will get this triangle:



On our problem, let's let $y = \arcsin u$. The definition of the inverse sine tells us that $\sin y = u$. Now let's take the derivative of both sides with respect to x using implicit differentiation:

$\frac{d}{dx}[\sin y] = \frac{du}{dx}$ You will need to use the chain rule on the left side.

$\cos y \cdot y' = u'$ Now solve for y'

$y' = \frac{u'}{\cos y}$ From our triangle we know that $\cos y = \frac{\sqrt{1-u^2}}{1}$. Plug this in for $\cos y$.

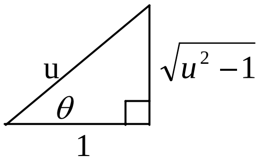
$y' = \frac{u'}{\frac{\sqrt{1-u^2}}{1}}$ Now simplify and we have our answer.

$y' = \frac{u'}{\sqrt{1-u^2}}$ So if $y = \arcsin u$ then $y' = \frac{u'}{\sqrt{1-u^2}}$.

EXAMPLE: If $y = arc\sec u$, find y'.

In order to do this we will draw a triangle like in the previous problems. We can rewrite this problem as $arc\sec\frac{u}{1}$. Then we know the hypotenuse is u and the adjacent side is 1.

Then we can use the Pythagorean Theorem to find the remaining side. You will get this triangle:



The definition of the inverse secant tells us that $\sec y = u$. Now let's take the derivative of both sides with respect to x using implicit differentiation:

$\frac{d}{dx}[\sec y] = \frac{du}{dx}$ You will need to use the chain rule on the left side.

$\sec y \tan y \cdot y' = u'$ Now solve for y'

$y' = \frac{u'}{\sec y \tan y}$ We know that $\sec y = u$ and $\tan y = \sqrt{u^2-1}$ from our triangle. Now substitute.

$y' = \frac{u'}{u\sqrt{u^2-1}}$ So if $y = arc\sec u$ then $y' = \frac{u'}{u\sqrt{u^2-1}}$.

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}\left[\arcsin u\right]=\frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}\left[\arccos u\right]=-\frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\left[\arctan u\right]=\frac{u'}{1+u^2} \qquad \frac{d}{dx}\left[arc\cot u\right]=-\frac{u'}{1+u^2}$$

$$\frac{d}{dx}\left[arc\sec u\right]=\frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}\left[arc\csc u\right]=-\frac{u'}{|u|\sqrt{u^2-1}}$$

EXAMPLE: Find the derivative: $f(x)=arc\sec(2x)$.

We will let $u=2x$. Then $u'=2$. We just need to substitute these into the formula $\frac{u'}{|u|\sqrt{u^2-1}}$. You will get:

$f'(x)=\frac{2}{|2x|\sqrt{(2x)^2-1}}$. Now we can simplify: $f'(x)=\frac{1}{|x|\sqrt{4x^2-1}}$. This is as far as we can go.

EXAMPLE: Find the derivative: $h(x)=x^2\arctan(x^2-3)$.

We need to use the product rule on this one. When we take the derivative of $\arctan(x^2-3)$ we will let $u=x^2-3$. Then $u'=2x$ and we will use the formula the formula $\frac{u'}{1+u^2}$. Using the product rule we get:

$h'(x)=x^2\cdot\frac{2x}{1+(x^2-3)^2}+\arctan(x^2-3)\cdot(2x)$. Now we can simplify:

$h'(x)=\frac{2x^3}{1+(x^2-3)^2}+2x\arctan(x^2-3)$. This is as far as we can go.

EXAMPLE: Find the derivative: $y=x\cdot\arctan(2x)-\frac{1}{4}\ln(1+4x^2)$.

We need to use the product rule on this one. When we take the derivative of $\arctan(2x)$ we will let $u=2x$. Then $u'=2$ and we will use the formula the formula $\frac{u'}{1+u^2}$. For the second term since we have a natural log we will let $u=1+4x^2$. Then $u'=8x$. To do a derivative of the ln we will use the formula $\frac{u'}{u}$. Using the product rule we get: $y'=x\cdot\frac{2}{1+(2x)^2}+\arctan(2x)\cdot(1)-\frac{1}{4}\cdot\frac{8x}{1+4x^2}$. Now we can simplify:

$y'=x\cdot\frac{2}{1+4x^2}+\arctan(2x)-\frac{2x}{1+4x^2}$. The first and last terms cancel and then we have: $y'=\arctan(2x)$

EXAMPLE: Find the derivative: $y = 2 \cdot \ln(\arcsin 3x)$.

Since we have a natural log we will let $u = \arcsin 3x$. Then $u' = \dfrac{3}{\sqrt{1-(3x)^2}}$. To do a derivative of the ln we will use the formula $\dfrac{u'}{u}$. This will give us: $y = 2 \cdot \dfrac{\frac{3}{\sqrt{1-9x^2}}}{\arcsin 3x}$. Now we can simplify:

$y' = \dfrac{6}{\arcsin(3x) \cdot \sqrt{1-9x^2}}$. Nothing more we can do on this one.

EXAMPLE: Find the derivative: $y = 25\arcsin\left(\dfrac{x}{5}\right) - x\sqrt{25-x^2}$. Write your answer as a single fraction.

For the first term we will let $u = \dfrac{x}{5}$. Then $u' = \dfrac{1}{5}$. You will use $\dfrac{u'}{\sqrt{1-u^2}}$. For the second term you will need to use the product rule combined with the chain rule. Putting this all together you will have:

$$y' = 25 \cdot \frac{\frac{1}{5}}{\sqrt{1-\left(\frac{x}{5}\right)^2}} - x \cdot \frac{1}{2}(25-x^2)^{-\frac{1}{2}}(-2x) - \sqrt{25-x^2}\,(1)$$

Now we need to simplify.

$$y' = \frac{5}{\sqrt{1-\frac{x^2}{25}}} + \frac{x^2}{\sqrt{25-x^2}} + \sqrt{25-x^2}$$

For $1 - \dfrac{x^2}{25}$ we can get common denominators: $\dfrac{25-x^2}{25}$.

$$y' = \frac{5}{\sqrt{\frac{25-x^2}{25}}} + \frac{x^2}{\sqrt{25-x^2}} + \sqrt{25-x^2}$$

This simplified further.

$$y' = \frac{5}{\frac{\sqrt{25-x^2}}{5}} + \frac{x^2}{\sqrt{25-x^2}} + \sqrt{25-x^2}$$

The first term can be simplified further.

$$y' = \frac{25}{\sqrt{25-x^2}} + \frac{x^2}{\sqrt{25-x^2}} + \frac{\sqrt{25-x^2}}{1}$$

Multiply the last term by $\dfrac{\sqrt{25-x^2}}{\sqrt{25-x^2}}$ to get common denominators.

$$y' = \frac{25 + x^2 - (25-x^2)}{\sqrt{25-x^2}}.$$

Now simplify the numerator and we are done.

$$y' = \frac{2x^2}{\sqrt{25-x^2}}.$$