6.3 Trigonometric Identities

This section will help you practice your trigonometric identities. We are going to establish an identity. What this means is to work out the problem and show that both sides of the identity are the same. First let's look at a list of identities we've already talked about plus a few more.

List of Identities

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ $\csc\theta = \dfrac{1}{\sin\theta}$ $\sec\theta = \dfrac{1}{\cos\theta}$ $\cot\theta = \dfrac{1}{\tan\theta}$

$\sin^2\theta + \cos^2\theta = 1$ $\sin^2\theta = 1 - \cos^2\theta$ $\cos^2\theta = 1 - \sin^2\theta$

$\sec^2\theta = 1 + \tan^2\theta$ $\tan^2\theta = \sec^2\theta - 1$ $\csc^2\theta = 1 + \cot^2\theta$ $\cot^2\theta = \csc^2\theta - 1$

EXAMPLE: Establish the identity: $\csc\theta \cdot \tan\theta = \sec\theta$.

You want to show that one side of the equation equals the other side. In these problems you are NOT allowed to do operations like adding or subtracting things from one side to the other. Think of each side as independent. We are not going to do anything with the right hand side. On the left side we will put in the identities for the cosecant and tangent functions. One technique is to change everything into sines and cosines:

$\dfrac{1}{\sin\theta} \cdot \dfrac{\sin\theta}{\cos\theta} = \sec\theta$ We can now cancel the sines from the left side of the equation.

$\dfrac{1}{\cos\theta} = \sec\theta$ We can change the fraction on the left side into secant.

$\sec\theta = \sec\theta$ Both sides are the same, so we are done.

EXAMPLE: Establish the identity: $\dfrac{\sin^4\theta - \cos^4\theta}{\cos\theta - \sin\theta} = -(\cos\theta + \sin\theta)$.

One of the first techniques you should try is factoring. We can factor the top because of difference of squares.

$\dfrac{(\sin^2\theta + \cos^2\theta)(\sin^2\theta - \cos^2\theta)}{\cos\theta - \sin\theta} = -(\cos\theta + \sin\theta)$ We know $\sin^2\theta + \cos^2\theta = 1$

$\dfrac{(\sin^2\theta - \cos^2\theta)}{\cos\theta - \sin\theta} = -(\cos\theta + \sin\theta)$ We can factor the top again by difference of squares.

$\dfrac{(\sin\theta - \cos\theta)(\sin\theta + \cos\theta)}{\cos\theta - \sin\theta} = -(\cos\theta + \sin\theta)$ We want to factor a negative out of the first term.

$\dfrac{-(-\sin\theta+\cos\theta)(\sin\theta+\cos\theta)}{\cos\theta-\sin\theta}=-(\cos\theta+\sin\theta)$ Now switch the order in the first term on top.

$\dfrac{-(\cos\theta-\sin\theta)(\sin\theta+\cos\theta)}{\cos\theta-\sin\theta}=-(\cos\theta+\sin\theta)$ Now we can cancel the $\cos\theta-\sin\theta$ terms.

$-(\sin\theta+\cos\theta)=-(\cos\theta+\sin\theta)$ Both sides are equal so the proof is done.

EXAMPLE: Establish the identity: $\dfrac{\cos x-2\sin x\cos x}{\cos^2 x-\sin^2 x+\sin x-1}=\cot x$.

$\dfrac{\cos x(1-2\sin x)}{\cos^2 x-\sin^2 x+\sin x-1}=\cot x$ First we can factor the numerator. Now we want to get all sines on the bottom. We can use the identity $\cos^2 x=1-\sin^2 x$.

$\dfrac{\cos x(1-2\sin x)}{(1-\sin^2 x)-\sin^2 x+\sin x-1}=\cot x$ Now simplify the denominator.

$\dfrac{\cos x(1-2\sin x)}{-2\sin^2 x+\sin x}=\cot x$ Factor the denominator.

$\dfrac{\cos x(1-2\sin x)}{\sin x(-2\sin x+1)}=\cot x$ The part in parenthesis on top and bottom can be cancelled.

$\dfrac{\cos x}{\sin x}=\cot x$ We will use the identity $\cot x=\dfrac{\cos x}{\sin x}$

$\cot x=\cot x$ Both sides are equal so we are done.

EXAMPLE: Establish the identity: $\cot\theta+\dfrac{1-2\cos^2\theta}{\sin\theta\cos\theta}=\tan\theta$.

Another technique for these kind of problems is to get common denominators if there are two separate fractions. At the same time I will also use the identity: $\cot\theta=\dfrac{\cos\theta}{\sin\theta}$.

$\dfrac{\cos\theta}{\sin\theta}\cdot\left(\dfrac{\cos\theta}{\cos\theta}\right)+\dfrac{1-2\cos^2\theta}{\sin\theta\cos\theta}=\tan\theta$ Now write as a single fraction.

$\dfrac{\cos^2\theta+1-2\cos^2\theta}{\sin\theta\cos\theta}=\tan\theta$ Now simplify the numerator.

$\dfrac{1-\cos^2\theta}{\sin\theta\cos\theta}=\tan\theta$ We will now use the identity $\sin^2\theta=1-\cos^2\theta$.

$\frac{\sin^2\theta}{\sin\theta\cos\theta}=\tan\theta$ We can cancel a sine from the top and bottom.

$\frac{\sin\theta}{\cos\theta}=\tan\theta$ We will use the identity $\tan\theta=\frac{\sin\theta}{\cos\theta}$.

$\tan\theta=\tan\theta$ Both sides are equal so we are done.

EXAMPLE: Establish the identity: $\frac{\cos x}{1+\sin x}+\frac{1+\sin x}{\cos x}=2\sec x$.

Once again we want to first get a single fraction so we need common denominators.

$\frac{\cos x}{1+\sin x}\cdot\left(\frac{\cos x}{\cos x}\right)+\frac{1+\sin x}{\cos x}\cdot\left(\frac{1+\sin x}{1+\sin x}\right)=2\sec x$ Now multiply and write as a single fraction.

$\frac{\cos^2 x+(1+\sin x)^2}{\cos x(1+\sin x)}=2\sec x$ We will expand the numerator.

$\frac{\cos^2 x+\sin^2 x+2\sin x+1}{\cos x(1+\sin x)}=2\sec x$ We will use the identity $\cos^2 x+\sin^2 x=1$

$\frac{1+2\sin x+1}{\cos x(1+\sin x)}=2\sec x$ Simplify the numerator.

$\frac{2\sin x+2}{\cos x(1+\sin x)}=2\sec x$ Factor the numerator.

$\frac{2(\sin x+1)}{\cos x(1+\sin x)}=2\sec x$ We can cancel the $\sin x+1$ from the top and bottom.

$\frac{2}{\cos x}=2\sec x$ We will use the identity $\sec x=\frac{1}{\cos x}$.

$2\sec x=2\sec x$ Both sides are the same, so we are done.

EXAMPLE: Establish the identity: $\frac{\tan x+\cot x}{\sec x\csc x}=1$.

Another technique is to change everything into sines and cosines. This makes it easier to reduce.

$$\frac{\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}}{\frac{1}{\cos x}\cdot\frac{1}{\sin x}}=1$$

We need to get common denominators in the numerator.

$$\frac{\frac{\sin x}{\cos x}\cdot\left(\frac{\sin x}{\sin x}\right)+\frac{\cos x}{\sin x}\cdot\left(\frac{\cos x}{\cos x}\right)}{\frac{1}{\cos x}\cdot\frac{1}{\sin x}}=1$$

Multiply and write as one fraction in the numerator.

$$\frac{\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}}{\frac{1}{\sin x\cos x}}=1$$

Now use the identity $\sin^2\theta+\cos^2\theta=1$.

$$\frac{\frac{1}{\sin x\cos x}}{\frac{1}{\sin x\cos x}}=1$$

Flip over the bottom fraction and multiply.

$$\frac{1}{\sin x\cos x}\cdot\frac{\sin x\cos x}{1}=1$$

We can cancel terms.

1 = 1

Both sides are the same so we are done.

EXAMPLE: Establish the identity: $\frac{1+\sin\theta}{\cos\theta}=\frac{\cos\theta}{1-\sin\theta}$.

We can't factor and they are both single fractions. We need to use a different technique. We are NOT allowed to cross multiply because we need to treat both sides separately. We will multiply one side by a conjugate.

$$\frac{1+\sin\theta}{\cos\theta}=\frac{\cos\theta}{1-\sin\theta}\cdot\left(\frac{1+\sin\theta}{1+\sin\theta}\right)$$

I chose the right side, but we can chose either side to work with.

$$\frac{1+\sin\theta}{\cos\theta}=\frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}$$

Now use the identity $\cos^2 x=1-\sin^2 x$.

$$\frac{1+\sin\theta}{\cos\theta}=\frac{\cos\theta(1+\sin\theta)}{\cos^2\theta}$$

Now we can cancel the cosine from the top and bottom.

$$\frac{1+\sin\theta}{\cos\theta}=\frac{1+\sin\theta}{\cos\theta}$$

Both sides are equal so we are done.

EXAMPLE: Establish the identity: $\dfrac{\sec^2 x - \tan^2 x + \tan x}{\sec x} = \sin x + \cos x$.

We will first start out by using the identity $\sec^2 x = \tan^2 x + 1$.

$\dfrac{(\tan^2 x + 1) - \tan^2 x + \tan x}{\sec x} = \sin x + \cos x$ Now simplify the numerator.

$\dfrac{1 + \tan x}{\sec x} = \sin x + \cos x$ Now change everything into sines and cosines.

$\dfrac{1 + \dfrac{\sin x}{\cos x}}{\dfrac{1}{\cos x}} = \sin x + \cos x$ We need common denominators on top.

$\dfrac{\left(\dfrac{\cos x}{\cos x}\right) + \dfrac{\sin x}{\cos x}}{\dfrac{1}{\cos x}} = \sin x + \cos x$ Combine as a single fraction on top.

$\dfrac{\dfrac{\cos x + \sin x}{\cos x}}{\dfrac{1}{\cos x}} = \sin x + \cos x$ Flip the bottom fraction and multiply.

$\dfrac{\cos x + \sin x}{\cos x} \cdot \dfrac{\cos x}{1} = \sin x + \cos x$ We can cancel the cosines.

$\cos x + \sin x = \sin x + \cos x$ Both sides are equal so we are done.

EXAMPLE: Establish the identity: $\dfrac{\cot x}{1 - \tan x} + \dfrac{\tan x}{1 - \cot x} = 1 + \tan x + \cot x$.

Sometimes you need to work out both sides, which we will do. I will first change both sides into sines and cosines.

$\dfrac{\dfrac{\cos x}{\sin x}}{1 - \dfrac{\sin x}{\cos x}} + \dfrac{\dfrac{\sin x}{\cos x}}{1 - \dfrac{\cos x}{\sin x}} = 1 + \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}$ We need common denominators on both sides.

$$\frac{\frac{\cos x}{\sin x}}{\left(\frac{\cos x}{\cos x}\right)-\frac{\sin x}{\cos x}}+\frac{\frac{\sin x}{\cos x}}{\left(\frac{\sin x}{\sin x}\right)-\frac{\cos x}{\sin x}}=\left(\frac{\sin x\cos x}{\sin x\cos x}\right)+\frac{\sin x}{\cos x}\cdot\left(\frac{\sin x}{\sin x}\right)+\frac{\cos x}{\sin x}\cdot\left(\frac{\cos x}{\cos x}\right)$$

Now combine.

$$\frac{\frac{\cos x}{\sin x}}{\frac{\cos x-\sin x}{\cos x}}+\frac{\frac{\sin x}{\cos x}}{\frac{\sin x-\cos x}{\sin x}}=\frac{\sin x\cos x+\sin^2 x+\cos^2 x}{\sin x\cos x}$$

We need to get rid of the double fractions.

$$\frac{\cos^2 x}{\sin x(\cos x-\sin x)}+\frac{\sin^2 x}{\cos x(\sin x-\cos x)}=\frac{\sin x\cos x+1}{\sin x\cos x}$$

We used the identity $\sin^2\theta+\cos^2\theta=1$.

I want to change the order of $\sin x-\cos x$. If I factor out a negative I will get $-(-\sin x+\cos x)$. This is equal to $-(\cos x-\sin x)$. Now I will put this into our problem, changing the plus sign in the middle to a negative.

$$\frac{\cos^2 x}{\sin x(\cos x-\sin x)}-\frac{\sin^2 x}{\cos x(\cos x-\sin x)}=\frac{\sin x\cos x+1}{\sin x\cos x}$$

We need common denominators.

$$\left(\frac{\cos x}{\cos x}\right)\cdot\frac{\cos^2 x}{\sin x(\cos x-\sin x)}-\left(\frac{\sin x}{\sin x}\right)\cdot\frac{\sin^2 x}{\cos x(\cos x-\sin x)}=\frac{\sin x\cos x+1}{\sin x\cos x}$$

Now multiply and combine.

$$\frac{\cos^3 x-\sin^3 x}{\sin x\cos x(\cos x-\sin x)}=\frac{\sin x\cos x+1}{\sin x\cos x}$$

Now we use difference of cubes formula.

$$\frac{(\cos x-\sin x)(\cos^2 x+\sin x\cos x+\sin^2 x)}{\sin x\cos x(\cos x-\sin x)}=\frac{\sin x\cos x+1}{\sin x\cos x}$$

Now we can cancel the $\cos x-\sin x$.

$$\frac{\cos^2 x+\sin x\cos x+\sin^2 x}{\sin x\cos x}=\frac{\sin x\cos x+1}{\sin x\cos x}$$

Now we use $\sin^2\theta+\cos^2\theta=1$.

$$\frac{\sin x\cos x+1}{\sin x\cos x}=\frac{\sin x\cos x+1}{\sin x\cos x}$$

Both sides are equal so we are finally done! Whew!