Final Exam

Differential Equations

December 15, 2000

Each problem is worth 20 points. You must show all of your work for full credit.

1. Solve the initial value problems.

a)

$$\frac{dy}{dx} = \frac{y-1}{x+3},\ y(-1) = 0$$

b)

$$x\frac{dy}{dx} = y + x^2,\ y(1) = 2$$

2.

a) Evaluate

$$\mathcal{L}\{t\cos 7t\}$$

b) Evaluate

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s-1)^2+4}\right\}$$

3. a) Given that $y_1 = e^x$ and $y_2 = x^2 + 2x + 2$ are solutions of

$$y'' - \frac{x+2}{x}y' + \frac{2}{x}y = 0,$$

find the largest interval where $\{y_1, y_2\}$ is a fundamental set of solutions (i.e., a basis) for the differential equations.

b) State the linearity property for the Laplace transform, $\mathcal{L}\{y\}$.

4. A large tank initially holds 400 L of pure water. A saltwater solution containing 3 kg of salt per liter flows in at a constant rate of 20 L/min. The solution in the tank is kept well stirred and flows out of the tank at the rate of 25 L/min. Find the mass of salt in the tank after t minutes (before the tank empties).

(note: as a realistic problem on the final, you would be given the governing differential equation model,

$$\begin{aligned}\frac{dy}{dt} &= \text{input rate - output rate} \\ &= 20\cdot 3 - 25\frac{y}{400-25t}\end{aligned}$$

5. Find the solution to the differential equation

$$9x^2y'' - xy' + y = 0,\ y(1) = \frac{17}{9},\ y'(1) = 1,\ x > 0$$

6. Find the general solution to $y'' - 4y = 1 + e^{2x}$

7. The motion of a spring-mass system is governed by

$$x''(t) + 8x'(t) + 25x(t) = 0.$$

Starting from equilibrium, the mass is pulled down 1 cm and released with a downward velocity of 2 cm/s.

a) State the initial conditions of the system and solve the initial value problem.
b) State whether the motion of the spring is underdamped, critically damped or overdamped.

9. Solve using LaPlace Transforms:

$$y'' + 2y' + 2y = 0,\ y(0) = 0,\ y'(0) = 3$$

10. Solve the system of equations.

$$\begin{aligned} x' &= y + \sin t,\ x(0) = 2 \\ y' &= x + 2\cos t,\ y(0) = 0 \end{aligned}$$