Linear Algebra Practice Problems
Final Exam, Fall 2007

1. Give an example of each of the following for the vector space $\mathbb{R}^3$:
a) A subspace.
b) A spanning set which is not linearly independent.
c) A linearly independent set of vectors which is not a spanning set.
d) A basis.

2. Find a basis for the subspace ${}_2\mathbb{R}_2$ spanned by

$$\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \right\}$$

3. Express the polynomial $t^2 + t + 2$ as a linear combination of the polynomials $\{(t-1)^2, t-1, 1\}$.

4. Given T defined on $\mathbb{R}^2$ by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 5x_1 + x_2 \\ 3x_1 - 2x_2 \end{bmatrix}$$

a) Confirm that this is a linear transformation.
b) Find the matrix of transformation, i.e., the matrix A such that $T(\vec{x}) = A\vec{x}$.

5. This problem uses the matrix

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$$

a) Find the eigenvalues.
b) Find the eigenvectors.
c) What is the geometric and algebraic multiplicities of each eigenvalue?
d) If possible, find a matrix S and a diagonal matrix D so that $S^{-1}AS = D$.
e) Is A defective?

6. Find the determinant.

$$B = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

7. Let W be a subspace of $\mathbb{R}^n$ with the spanning set

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 10 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \\ 4 \end{bmatrix} \right\}$$

Find a subset of S that is a basis for W.

8. Let $U = \{p(x) \in \mathbb{P}_3 : \ p(1) = 1\}$. Is U a vector space?

9. Which of the following statements are true?
a) Given that a vector space $\mathcal{V}$ has dimension n, any set of n linearly independent vectors in $\mathcal{V}$ is a spanning set of $\mathcal{V}$.
b) Given W a subspace of vector space $\mathcal{V}$. If $\{\vec{u}, \vec{v}, \vec{w}\}$ is a linearly independent subset of W, then $\{\vec{u}, \vec{v}\}$ is also linearly independent.
c) Given W a subspace of vector space $\mathcal{V}$. If $\dim(W) = m$, then every set of m linearly independent vectors in W is a basis for W.