Polynomial Particular Solutions for Poisson Problems

A.S. Muleshkov
Department of Mathematical Sciences
University of Nevada, Las Vegas, U.S.A

M.A. Golberg
517 Bianca Bay Street, Las Vegas, U.S.A.

A. H.-D. Cheng
Department of Civil Engineering
University of Mississippi
U.S.A.

C.S. Chen
Department of Mathematical Sciences
University of Nevada, Las Vegas, U.S.A.

Polynomial Interpolation

Consider a rectangular region $\bar{\Omega} = [a, b] \times [c, d]$ containing Ω in R^2 and let f in (8) be extendable continuously to $\hat{\Omega}$. Then, it is well known that f can be approximated by a Chebyshev polynomial of the form

$$q_{m,n}(x, y) = \sum_{k=0}^{m} \sum_{j=0}^{n} a_{jk} T_j \left(\frac{ax - a - b}{b - a} \right) T_k \left(\frac{2y - c - d}{c - d} \right) \tag{1}$$

where

$$a_{kj} = \frac{1}{nm\bar{c}_j\bar{c}_k} \sum_{q=0}^{m} \sum_{p=0}^{n} f(x_p, y_q) \cos\left(\frac{\pi pj}{n}\right) \cos\left(\frac{\pi qk}{m}\right), \tag{2}$$

and

$$x_p = \left(\frac{b - a}{2}\right) \cos\left(\frac{p\pi}{n}\right) + \frac{a + b}{2}, 0 \leq p \leq n \tag{3}$$

$$y_q = \left(\frac{d - c}{2}\right) \cos\left(\frac{q\pi}{n}\right) + \frac{c + d}{2}, 0 \leq q \leq n \tag{4}$$

Similarly, if $\hat{\Omega} = [a,b] \times [c,d] \times [g,h] \supseteq \Omega$ in R^3 , then f can be approximated by

$$q_{l,m,n}(x,y,z) \tag{5}$$

$$= \sum_{j=0}^{l} \sum_{k=0}^{m} \sum_{\ell=0}^{n} a_{jk\ell} T_j \left(\frac{ax - a - b}{b - a} \right) T_k \left(\frac{2y - c - d}{c - d} \right) T_\ell \left(\frac{2z - e - g}{e - g} \right) \tag{6}$$

where

$$a_{jk\ell} = \frac{1}{lmn\bar{c}_j\bar{c}_k\bar{c}_\ell} \sum_{r=0}^{\int} \sum_{s=0}^{m} \sum_{t=0}^{n} f(x_r, y_s, z_t) \cos\left(\frac{\pi r j}{l}\right) \cos\left(\frac{\pi s k}{m}\right) \cos\left(\frac{\pi t \ell}{n}\right). \tag{7}$$

and where (x_r, y_s, z_t) are defined analogously to (3) - (4).

Here, $T_n(x) = \cos(n \arccos x)$ is the Chebishev polynomial of degree n and $\bar{c}_j = 1/2$, if $j = 0$ or $j = n$ and $\bar{c}_j = 0$ otherwise. Similarly, $\bar{c}_k$ is found. It can be shown that $q_{n,m}(x,y)$ and $q_{l,m,n}(x,y,z)$ converge spectrally to f , if f is sufficiently smooth.

The Spliting Method

Let L be a linear partial differential operator, f - a given function and Ω - a bounded subset of $R^d, d = 2, 3$, with a boundary S. We consider the boundary value problem

$$Lu(P) = f(P), \quad P \in \Omega, \tag{8}$$

$$Bu(P) = g(P), \quad P \in \partial\Omega, \tag{9}$$

where B is a given boundary operator and g is a given boundary value.

In this approach, we begin by defining a particular solution, u_p, which satisfies (8) but not necessarily (9). Then letting

$$v = u - u_p. \tag{10}$$

v satisfies the homogeneous boundary value problem

$$Lv(P) = 0, \quad P \in \Omega, \tag{11}$$

$$Bv(P) = f(P) - Bu_p, \quad P \in \partial\Omega. \tag{12}$$

Let $\{\phi_k\}_1^k$ be a set of basis functions and approximate f by $\hat{f}$ where

$$\hat{f} = \sum_{k=1}^{N} a_k \phi_k. \tag{13}$$

Then, an "approximate" particular solution, $\hat{u}_p$, is defined by

$$\hat{u}_p = \sum_{k=1}^{N} a_k \Phi_k \tag{14}$$

where $\{\Phi_k\}_1^N$ are obtained by analytically solving

$$L\Phi_k = \phi_k. \tag{15}$$

By linearity, it follows that

$$L\hat{u}_p = \hat{f} \tag{16}$$

and $\hat{u}_p$ is used instead of u_p in subsequent calculations.

Evaluation of Particular Solutions

The 2D Case

A particular solution of

$$\Delta\psi = x^n y^m, \quad m \geq 0, n \geq 0, \tag{17}$$

is given by [Cheng et al, 1994]

$$\psi(x,y) = \begin{cases} \displaystyle\sum_{k=1}^{[\frac{n+2}{2}]} (-1)^{k+1} \frac{m!n!x^{m+2k}y^{n-2k+2}}{(m+2k)!(n-2k+2)!}, & \text{for } m \geq n, \\ \displaystyle\sum_{k=1}^{[\frac{m+2}{2}]} (-1)^{k+1} \frac{m!n!x^{m-2k+2}y^{n+2k}}{(m-2k+2)!(n+2k)!} & \text{for } m < n, \end{cases} \tag{18}$$

where $[x]$ is the largest integer that is less than or equal to x. Chen et al. (2000) gave a different approach for deriving particular solutions when the right-hand side of (17) is a homogeneous polynomial of degree n.

Theorem 1 *A particular solution of*

$$\Delta\psi = \sum_{k=0}^{n} A_k x^{n-k} y^k \tag{19}$$

is given by

$$\psi = \sum_{k=0}^{n} P_k x^{n-k+2} y^k \tag{20}$$

where

$$P_k = \sum_{m=0}^{[\frac{n-k}{2}]} \frac{(-1)^m (k+2m)!(n-k-2m)!}{k!(n-k+2)!} A_{k+2m}, \ \ \textit{for}\ 1 \le k \le n. \tag{21}$$

The 3D Case

A particular solution of

$$\Delta\psi = x^l y^m z^n \tag{22}$$

can be obtained by inspection similar to the 2D case:

$$\psi = \sum_{i=1}^{[\frac{m}{2}]+[\frac{n}{2}]+1} x^{l+2i} \sum_{j=\max\left\{1,i-[\frac{n}{2}]\right\}}^{\min\left\{i,[\frac{m}{2}]+1\right\}} a_{ij} y^{m-2j+2} z^{n-2i+2j} \tag{23}$$

where

$$\begin{aligned} a_{ij} &= \frac{-1}{(l+2i)(l+2i-1)}\left[(m-2j+4)(m-2j+3)a_{i-1,j-1}\right. \\ &\quad \left. + (n-2i+2j+2)(n-2i+2j+1)a_{i-1,j}\right], \end{aligned} \tag{24}$$

$$a_{11} = \frac{-1}{(l+2)(l+1)}, \quad a_{i0} = a_{0j} = 0. \tag{25}$$

An alternative approach is to find a particular solution when the right-hand side is a homogeneous polynomial of degree n as we have shown in the 2D case.

Let $Q_n(x, y, z)$ be an arbitrary homogeneous polynomial of power $n > 0$; i.e.,

$$Q_n(x, y, z) = \sum_{k=0}^{n} z^k \sum_{m=0}^{n-k} A_{k,m} x^{n-k-m} y^m. \tag{26}$$

For the equation

$$\Delta\psi = Q_n(x, y, z), \tag{27}$$

we are looking for a particular solution in the form

$$\psi = \sum_{k=0}^{n+2} z^k \sum_{m=0}^{n-k+2} P_{k,m} x^{n-k-m} y^m. \tag{28}$$

Substituting (28) into (27) and comparing the coefficients of $\Delta\psi$ with Q_n, we obtain the following system of equations:

$$\begin{aligned} A_{k,m} = \; & (n-k-m+2)(n-k-m+1)P_{k,m} + (m+2)(m+1)P_{k,m+2} \\ & +(k+2)(k+1)P_{k+2,m}, \quad k \geq 0, \quad m \geq 0, \quad m+k \leq n. \end{aligned} \tag{29}$$

The above system contains $(n+1)(n+2)/2$ equations with $(n+3)(n+4)/2$ unknowns. This means that there are $2n+5$ free parameters which can be set to zero due to the fact that the particular solution is not unique. The unknowns $P_{k,n}$ in (29) can be partitioned into $[(n+4)/2]$ subsets as follows:

$$\begin{aligned} S_{[\frac{n+2}{2}]} &= \{P_{k,m} : k+m = n+2 \text{ or } m+m = n+1\} \\ S_{[\frac{n}{2}]} &= \{P_{k,m} : k+m = n \text{ or } m+m = n-1\} \\ S_{[\frac{n-2}{2}]} &= \{P_{k,m} : k+m = n-2 \text{ or } m+m = n-3\} \\ &\vdots \end{aligned}$$

If n is odd, then $S_0 = \{P_{0,0}, P_{0,1}, P_{1,0}\}$. If n is even, then $S_0 = \{P_{0,0}\}$. Since $S_{[\frac{n+2}{2}]}$ contains $2n+5$ elements, all of these elements can be set to zero. In (29), we observe that S_j, $0 \leq j \leq [\frac{n}{2}]$, can be expressed in terms of the elements of S_{j+1}. We summarize the above stated procedure into the following algorithm.

Algorithm for evaluating $P_{k,m}$ $(k \geq 0, m \geq 0, k + m = l \leq n + 2):$

```
INPUT $n$ (degree of the homogeneous polynomial)
INPUT $A_{k,m}$, $k = 0, \cdots, n$; $m = 0, \cdots, n$.
Step 1 For $k = 0, \cdots, n+2$ set $P_{k,n-k+2} = 0$
Step 2 For $k = 0, \cdots, n+1$ set $P_{k,n-k+1} = 0$
Step 3 For $l = n, \cdots, 0$, For $k = 0, \cdots, l$ set $m = l - k$ and
           $P_{k,m} = \frac{A_{k,m} - (m+2)(m+1)P_{k,m+2} - (k+2)(k+1)P_{k+2,m}}{(n-l+2)(n-l+1)}$
OUTPUT For $k = 0, \cdots, n+2; m = 0, \cdots, n-k+2$, Print $P_{k,m}$.
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Remark: The above algorithm does not work for degree zero. For convenience, in the case of a particular solution of degree zero, $Q_0 = A_{0,0,}$ we use the particular solution $A_{0,0}x^2/2$.

Our algorithm for finding polynomial particular solutions for Poisson's equation consists of the following steps:

1. approximate f by using (1) - (2);
2. expand the interpolants in monomial form using a symbolic language such as Mathematica;
3. use (20) - (21) or the algorithm given in this section to obtain particular solutions for homogeneous polynomials and then add the results.

Finally, to solve the homogeneous Laplace equation, we use the Method of Fundamental Solutions (MFS).

Numerical Example

Consider the 2D Poisson problem

$$\Delta u(x,y) = f(x,y), \quad (x,y) \in \Omega, \tag{30}$$

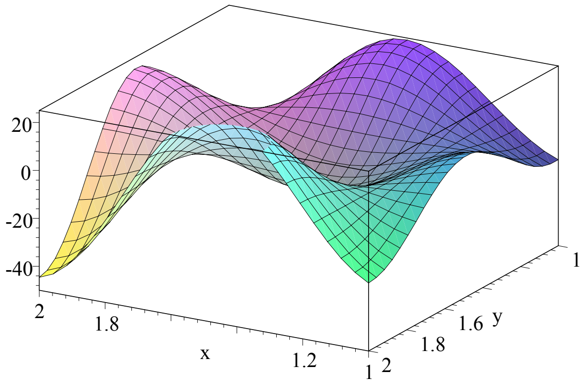
$$u(x,y) = g(x,y), \quad (x,y) \in \partial\Omega, \tag{31}$$

where $\Omega = [1,2]^2$ and

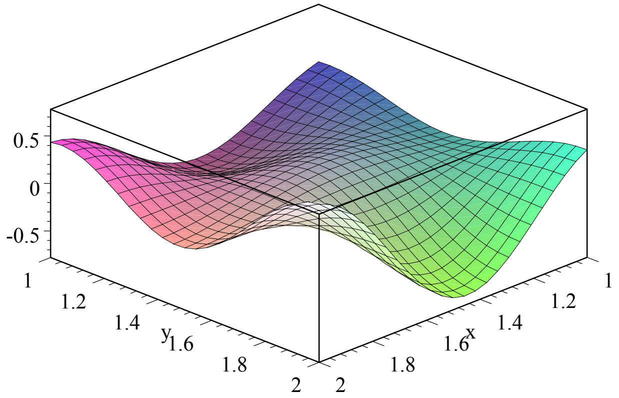
$$\begin{aligned} f(x,y) &= -\frac{751\pi^2}{144} \sin\frac{\pi x}{6} \sin\frac{7\pi x}{4} \sin\frac{3\pi y}{4} \sin\frac{5\pi y}{4} + \frac{7\pi^2}{12} \cos\frac{\pi x}{6} \cos\frac{7\pi x}{4} \\ &\times \sin\frac{3\pi y}{4} \sin\frac{5\pi y}{4} + \frac{15\pi^2}{8} \sin\frac{\pi x}{6} \sin\frac{7\pi x}{4} \cos\frac{3\pi y}{4} \cos\frac{5\pi y}{4}, \\ g(x,y) &= \sin\frac{\pi x}{6} \sin\frac{7\pi x}{4} \sin\frac{3\pi y}{4} \sin\frac{5\pi y}{4}, \end{aligned}$$

The exact solution is

$$u(x,y) = \sin\frac{\pi x}{6} \sin\frac{7\pi x}{4} \sin\frac{3\pi y}{4} \sin\frac{5\pi y}{4}.$$



Forcing term



Exact Solution

To approximate the forcing term using Chebyshev interpolation, we choose $m = n$ in (1) since the solution domain is a square. We note that the particular solution is not unique. In this example, the approximate particular solution $u_p(x, y)$ was obtained by taking $n = 7$. The MFS was applied to evaluate the homogeneous solution. In the MFS, we choose 40 uniformly distributed collocation points on the boundary. The same number of source points on the fictitious boundary, a circle with radius 2 and center $(1.5, 1.5)$, were also chosen. The approximate solutions $\hat{u}$ are evaluated at six randomly chosen points. The numerical results in Table 1 are in excellent agreement with the exact solution.

x	y	$\|u - \hat{u}\|_\infty$
1.2	1.3	5.605×10^{-5}
1.2	1.4	1.408×10^{-6}
1.3	1.5	1.552×10^{-4}
1.5	1.8	2.558×10^{-4}
1.4	1.4	1.052×10^{-4}
1.6	1.7	1.673×10^{-4}

Table 1: The errors of $u - \hat{u}$ at six points.