Polynomial Particular Solutions for Certain Partial Differential Operators

M. A. Golberg,[1] **A. S. Muleshkov,**[2] **C. S. Chen,**[2] **A. H.-D. Cheng**[3]
[1]*517 Bianca Bay Street, Las Vegas, Nevada 89144*

[2]*Department of Mathematical Sciences, University of Nevada, Las Vegas, Nevada 89154-4020*

[3]*Department of Civil Engineering, University of Mississippi, University, Mississippi 38677*

Received 23 December 2001; accepted 12 July 2002

DOI 10.1002/num.10033

In this article, we consider a variant of the Dual Reciprocity Method (DRM) for solving boundary value problems based on approximating source terms by polynomials other than the traditional basis functions. The use of pseudo-spectral approximations and symbolic methods enables us to obtain highly accurate results without solving the often ill-conditioned equations that occur when radial basis function approximations are used. When the given partial differential equation is either Poisson's equation or an inhomogeneous Helmholtz-type equation, we are able to obtain either closed form particular solutions or efficient recursive algorithms. Using the particular solutions, we convert the inhomogeneous equations to homogeneous. The resulting homogeneous equations are then amenable to solution by boundary-type methods such as the Boundary Element Method (BEM) or the Method of Fundamental Solutions (MFS). Using the MFS, we provide numerical solutions to a variety of boundary value problems in $\mathbf{R}^2$ and $\mathbf{R}^3$. Using this approach, we can achieve high accuracy with a modest number of interpolation and collocation points. © 2002 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 19: 112–133, 2003

Keywords: particular solution; method of fundamental solutions; radial basis functions; Chebyshev polynomial; multidimensional interpolation

I. INTRODUCTION

Since the introduction of the Dual Reciprocity Method (DRM) by Nardini and Brebbia [1], there has been continuing interest in calculating particular solutions for various partial differential operators. Most commonly, this has been done for the Laplacian, where the source term is approximated by a series of radial basis functions (RBFs) [2]. In this case, one can often obtain

Correspondence to: C. S. Chen, Department of Mathematical Sciences, University of Nevada, Las Vegas, NV 89154–4020 (e-mail: chen@unlv.edu)

Contract grant sponsor: NATO Collaborative Linkage Grant; contract grant number: PST.CLG.977633 (to C. S. Chen)

analytic formulas, because the computation can be reduced to determining particular solutions for the one-dimensional (1D) radial part of the operator. For other operators, such as Helmholtz-type operators, this task has proved to be more difficult. For example, in [3], Muleshkov et al. obtained particular solutions for Helmholtz-type operators when the source terms were polyharmonic splines. The procedure given in [3] can be generalized to operators that are products of Helmholtz-type and Laplace operators. These analytic results have been obtained because of the fortuitous matching of the radial symmetry of these operators to the radial symmetry of the RBF's. Traditionally, RBFs have been used to approximate source terms, because the standard formulation of the DRM using Green's Theorem requires one to use collocation points in the physical domain of the given boundary value problem. Because interpolation has been the method of choice for the approximation of arbitrary source terms, one is generally required to choose the interpolation points in the physical domain. Hence, there is the need for interpolants that can be obtained using arbitrary scattered data. In this respect, RBFs play a pivotal role, because they are the only known class of globally defined functions with this property. For example, due to a theorem of Haar [4], this is generally not possible for polynomials and trigonometric functions.

Despite the important interpolating properties of the RBFs, there are some drawbacks regarding their use as approximation functions. For example, it is difficult to obtain rapidly convergent RBF interpolants, because the best known bases for doing this, the multiquadrics and Gaussians, each contain unknown shape parameters whose optimal values are difficult or impossible to obtain. As a consequence, one often has to use a large number of interpolating points, and this requires the solution of large, highly ill-conditioned systems of equations. Because of these difficulties, other classes of approximants were considered in the past. For example, Atkinson [5] and Cheng et al. [6] examined the use of trigonometric and polynomial approximations for Poisson's equation. Atkinson did not obtain analytical formulas for the case of polynomials. The trigonometric approximations in [5] were computationally intensive. However, because polynomials have many useful properties, in this article, we reconsider their use for approximating source terms, and we show how to overcome some of the difficulties encountered previously.

First, rather than using Green's theorem, we will implement the DRM as the MPS (method of particular solutions). This allows one to decouple the calculation of particular solutions from the boundary method used to solve the resulting homogeneous PDE. As a consequence, we do not require collocation points only in the physical domain. For example, as we show in [7], they can be chosen as points on a regular rectangular grid in $\mathbf{R}^2$ or a box containing the physical domain in $\mathbf{R}^3$. This is analogous to the classical spectral method for solving PDEs, which is based on tensor product Chebyshev interpolants. Because explicit formulas are known for these interpolants, there is no need to solve the system of linear equations. Hence, the difficulties observed by previous authors are not present in our approach. Second, for several important classes of operators such as the Laplacian and the Helmholtz operator, we are able to obtain either explicit formulas or efficient algorithms for finding analytic particular solutions based on these polynomial approximations.

The article is organized as follows: In Section II, we show how a number of inhomogeneous and time-dependent problems can be solved by using the MPS rather than the standard DRM, which enables us to use tensor product interpolants rather than scattered data interpolants required by the standard DRM. In Section III, we review some well known properties of tensor product Chebyshev interpolants. In Section IV, we obtain an efficient algorithm for calculating particular solutions for the Laplacian in 3D, which extends the formula given by Cheng et al. [6] for the Laplacian in 2D. In Section V, we give an explicit formula for a particular solution for Helmholtz-type operators with a monomial right-hand side. Coupling this with a symbolic algorithm for converting Chebyshev polynomials to monomial form enables us to obtain approximate particular solutions. In Section VI, we review the method of fundamental solutions, which we use to solve the related homogeneous equations. In Section VII, we provide some numerical examples to establish the efficiency of this approach for solving various boundary value problems (BVPs). We conclude with some discussion of future research.

II. THE METHOD OF PARTICULAR SOLUTIONS

A. Elliptic Problems

Let L be a linear partial differential operator; f, a given function; and Ω, a bounded subset of $\mathbf{R}^{\mathbf{d}}$, $d = 2, 3$, with boundary $\partial\Omega$. We consider the boundary value problem

$$Lu(P) = f(P), \qquad P \in \Omega, \tag{2.1}$$

$$Bu(P) = g(P), \qquad P \in \partial\Omega, \tag{2.2}$$

where B is a given boundary operator and g is a given function. A standard method for solving (2.1)–(2.2) numerically is to convert them to an equivalent integral equation under the assumption that L has a known fundamental solution, $G(P, Q)$. This process usually produces a domain integral term of the form

$$\int_{\Omega} G(P, Q) f(P) dV. \tag{2.3}$$

As is well known [2], the evaluation of (2.3) can be difficult because G is singular and the domain Ω can be arbitrary. In this case, the use of standard numerical methods is often the most costly part of the calculation. Because of this, there has been considerable research done over the past two decades to avoid the direct calculation of the volume integral (2.3). Perhaps the most commonly used technique in the engineering literature is the DRM, where the source term f is approximated and the domain integral, (2.3), is reduced to one over the boundary $\partial\Omega$ by repeated use of Green's Theorem. In this procedure, the resulting boundary integral equation is coupled to the method of approximating f through the common use of collocation and interpolation points [2]. This coupling makes if difficult to use standard approximating functions such as polynomials and trigonometric functions, because scattered data interpolation usually fails [4]. To overcome this problem, we consider the alternative and often equivalent method of particular solutions.

In this approach, we begin by defining a particular solution, u_p, which satisfies (2.1) but not necessarily (2.2). Then letting

$$v = u - u_p. \tag{2.4}$$

v satisfies the homogeneous boundary value problem

$$Lv(P) = 0, \qquad P \in \Omega, \tag{2.5}$$

$$Bv(P) = f(P) - Bu_p, \qquad P \in \partial\Omega. \tag{2.6}$$

In this case, equivalent boundary integral equations do not contain the domain integral, (2.3), and so they could be solved by well-known numerical techniques. Other, boundary-only, procedures, such as the method of fundamental solutions can also be used [9]. Hence, the remaining problem is to obtain an appropriate particular solution. Because the domain integral, (2.3), is a particular solution, it could, in principle, be used, but, as we have already observed, it is often desirable to avoid its direct computation.

One method for avoiding this difficulty was proposed by Atkinson in 1985 [5]. In that article, he observed that if f could be extended smoothly to a domain $\hat{\Omega}$ containing Ω, then the integral

$$\int_{\hat{\Omega}} G(P, Q) f dV \tag{2.7}$$

is also a particular solution. By choosing $\hat{\Omega}$ to be an ellipse in $\mathbf{R}^2$ or an ellipsoid in $\mathbf{R}^3$, (2.7) can be evaluated by standard product integration rules [5]. This procedure has the advantage that it is quite general and requires no meshing of either Ω or $\hat{\Omega}$. However, computing the integral (2.7) can be quite time consuming. To the best of our knowledge, this approach has been used only in a few selected cases [8]. In most engineering work, the direct evaluation of (2.3) or (2.7) is avoided by using the following approach: Let $\{\phi_k\}_{k=1}^N$ be a set of basis functions and approximate f by $\hat{f}$ where

$$\hat{f} = \sum_{k=1}^{N} a_k \phi_k. \tag{2.8}$$

(Usually, a_k are determined by interpolation [1–3, 7], but this does not need to be the case.) Then, an "approximate" particular solution, $\hat{u}_p$, is defined by

$$\hat{u}_p = \sum_{k=1}^{N} a_k \Phi_k \tag{2.9}$$

where $\{\Phi_k\}_{k=1}^N$ are obtained by analytically solving

$$L\Phi_k = \phi_k, \qquad k = 1, 2, \dots, N. \tag{2.10}$$

By linearity, it follows that

$$L\hat{u}_p = \hat{f} \tag{2.11}$$

and $\hat{u}_p$ is used instead of u_p in subsequent calculations.

For this procedure, to be efficient, one needs to choose ϕ_k to accurately approximate f and be such that Φ_k can be obtained analytically. In many applications, $L = \Delta$ and good choices are RBFs such as splines and multiquadrics. As we have already mentioned, in the usual DRM approach, the coefficients a_k are obtained by using interpolation with the interpolation points in the physical region $\Omega \cup \partial\Omega$. However, in the method of particular solutions, this is not necessary, and one can use interpolation points outside of the physical region. Also, one does not have to be restricted to using RBFs, because the interpolation points can now be chosen on a regular mesh. This enables one to avoid the difficulty of scattered data approximation that RBFs are designed to solve. As a consequence, it becomes possible to use polynomial interpolants, which have the advantage of being high-order approximations. This is difficult to do with RBFs, because the available bases, such as MQs and Gaussians each have an unknown shape parameter whose appropriate value is difficult to find. Moreover, as is well known, it is possible to obtain polynomial interpolants explicitly without solving linear equations that are often ill-conditioned in RBF interpolation. This will be shown in the following section. Hence, to determine $\hat{u}_p$, we need to be able to solve (2.10) when $\{\phi_k\}$ form an appropriate polynomial basis and L is a given partial differential operator. In Section IV, we show how to do this when L is a Laplacian and, in Section V, when L is a Helmholtz-type operator.

B. Time-Dependent Problems

Although the MPS has been most often used to solve elliptic problems, its use can be extended to solve various of these problems by reducing the solution of time-dependent problems to solving a set of elliptic ones. As an example, consider the initial-BVP,

$$Lu(P, t) = \frac{\partial u(P, t)}{\partial t}, \qquad P \in \Omega,\ t > 0, \tag{2.12}$$

$$Bu(P, t) = g(P, t), \qquad P \in \partial\Omega,\ t > 0, \tag{2.13}$$

$$u(P, 0) = h(P), \qquad P \in \Omega \cup \partial\Omega, \tag{2.14}$$

where g and h are given functions and L, Ω, and $\partial\Omega$ are as in Section A. A standard method for solving (2.12)–(2.14) is to use a finite difference approximation for $\partial u/\partial t$ that reduces the problem to solving a sequence of elliptic BVP's. For example, suppose we want to solve (2.12)–(2.14) for $0 < t \leq T$. Let $\tau = T/n$ and approximate $\partial u/\partial t$ by

$$\frac{\partial u(P, n\tau)}{\partial t} \simeq \frac{u(P, n\tau) - u(P, (n-1)\tau)}{\tau}. \tag{2.15}$$

Letting

$$u_n(P) = u(P, n\tau) \tag{2.16}$$

and using (2.16) in (2.12), the resulting approximation $V_n(P)$ to $u_n(P)$ satisfies

$$LV_n(P) = \frac{V_n(P) - V_{n-1}(P)}{\tau}, \qquad P \in \Omega, \tag{2.17}$$

$$BV_n(P) = g(P, n\tau) \equiv g_n(P), \qquad P \in \partial\Omega, \tag{2.18}$$

$$V_0(P) = h(P). \tag{2.19}$$

One can see that (2.17) and (2.18) constitute a sequence of elliptic problems of the form (2.1)–(2.2) with the operator L replaced by the operator $L - 1/\tau$ and $f = -V_{n-1}/\tau$. (When $L =$

Δ, this method is commonly referred to as Rothe's method [10].) For each n, one can now solve (2.17)–(2.18) by using the MPS as indicated in Section A, provided that one can find particular solutions for $L - \lambda^2$, where $\lambda = 1/\sqrt{\tau}$. In the important case when $L = \Delta$, we arrive at the modified Helmholtz operator, whose particular solutions with a polynomial right-hand side are obtained in Section V. It is important to note that many other time-dependent problems, such as those for the wave equation, convection-diffusion equation, and nonlinear reaction-diffusion equations can also be reduced to solving BVP's for the modified Helmholtz equation [11].

III. POLYNOMIAL INTERPOLATION

As is well known, if $x_0 < x_1 < \cdots < x_n$ are $n + 1$ distinct points in $\mathbf{R}$, there exists a unique polynomial of degree $\leq n$ that interpolates a function f defined on $[x_0, x_n]$. More precisely, if p_n denotes the interpolating polynomial, then

$$p_n(x_k) = f(x_k), \qquad 0 \leq k \leq n. \tag{3.1}$$

Letting $l_j(x)$ be the unique polynomial satisfying

$$l_j(x_k) = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{3.2}$$

then $p_n(x)$ can be written in the form

$$p_n(x) = \sum_{j=0}^{n} f(x_j) l_j(x). \tag{3.3}$$

Equation (3.3) is usually called the Lagrange form of p_n and $\{l_j\}_{j=0}^n$, the *fundamental polynomials of Lagrange interpolation.* It is easily shown that l_j, $0 \leq j \leq n$, are given explicitly by

$$l_j(x) = \frac{\prod\limits_{k=0, k\neq j}^{n} (x - x_k)}{\prod\limits_{k=0, k\neq j}^{n} (x_j - x_k)}. \tag{3.4}$$

Although the existence of $p_n(x)$ requires only that the interpolation points be distinct, generally one imposes additional conditions on $\{x_k\}_{k=0}^n$ in order to guarantee that the sequence $p_n(x)$, $n \geq 0$ converges to $f(x)$ in some sense. For example, it is well known that choosing $\{x_k\}_{k=0}^n$ to be equally spaced, then $p_n(x)$ will not converge uniformly for all continuous functions f [4]. To guarantee convergence for sufficiently smooth f's, it suffices to choose $\{x_k\}_{k=0}^n$ as the zeros of a sequence of orthogonal polynomials of degree $n + 1$ corresponding to a non-negative integrable weight function [4]. For example, if we let $q_{n+1}(x) = T_{n+1}(x) = \cos((n + 1)\cos^{-1}x)$, $-1 \leq x \leq 1$, the $(n + 1)$st Chebyshev polynomial, then $T_{n+1}(x)$ has $n + 1$ distinct zeros

$$x_k = \cos\left[\frac{(2k+1)\pi}{2(n+1)}\right], 0 \le k \le n, \tag{3.5}$$

and the resulting sequence of interpolating polynomials converges uniformly to f, $-1 \le x \le 1$, if f is continuously differentiable [4].

Although $\{x_k\}_{k=0}^n$ given by (3.5) are good interpolation points in $[-1, 1]$, for our purposes, it is convenient to use the pseudo-spectral points

$$x_k = \cos\left(\frac{k\pi}{n}\right), \qquad 0 \le k \le n, \tag{3.6}$$

which include the end points $\{-1, 1\}$. (These are also called the Gauss-Lobatto points [12].) One can then show that, in this case, [13, 14]

$$l_j(x) = \frac{(1-x^2)T_n'(x)(-1)^{j+1}}{\bar{c}_j n^2(x-x_j)}, \qquad 0 \le j \le n, \tag{3.7}$$

where $\bar{c}_0 = \bar{c}_n = 2$ and $\bar{c}_j = 1$, $1 \le j \le n-1$. Consequently, it can be shown that [13, 14]

$$p_n(x) = \sum_{k=0}^{n} a_k T_k(x) \tag{3.8}$$

where

$$a_k = \frac{2}{n\bar{c}_k} \sum_{j=0}^{n} \frac{f(x_j)}{\bar{c}_j} \cos\left(\frac{\pi jk}{n}\right), \qquad 0 \le k \le n. \tag{3.9}$$

To use these formulas on an arbitrary interval $[a, b]$, we observe that $a \le x \le b$ can be given by

$$x = \alpha\xi + \beta, \qquad \alpha = \frac{b-a}{2}, \qquad \beta = \frac{a+b}{2}, \tag{3.10}$$

where $-1 \le \xi \le 1$. Then, the interpolating polynomial $q_n(x)$ for $x_k \in [a, b]$, $0 \le k \le n$, can be written as

$$q_n(x) = \sum_{k=0}^{n} a_k T_k\left(\frac{2x-b-a}{b-a}\right), \tag{3.11}$$

where a_k is the same as in (3.9) and x_j becomes $x_j = \alpha\xi_j + \beta$.

As for Chebyshev interpolants on the classical Chebyshev nodes, the pseudo-spectral interpolant $q_n(x)$ given by (3.8)–(3.9) can provide a spectral approximation to f. In fact, it was shown in [15] that if $f \in C^s[-1, 1]$, $s \ge 1$,

$$\|f - q_n\|_\ell \leq cn^{2\ell - s}\|f\|_s, \qquad 0 \leq \ell \leq \left[\frac{s}{2}\right]$$

where

$$\|f\|^2 = \int_{-1}^{1} \frac{f^2(x)}{\sqrt{1 - x^2}}\, dx$$

and

$$\|f\|_\ell^2 = \|f\|^2 + \|f'\|^2 + \cdots + \|f^{(\ell)}\|^2.$$

In particular,

$$\|f - q_n\|^2 \leq cn^{-s}\|f\|_s,$$

so that the convergence of q_n to f is spectral along with the first $[s/2]$ derivatives. Using a scaling argument as above, if $f \in C^s[a, b]$, $\{q_n\}$ converges spectrally to f.

A. Multidimensional Interpolation

Although one can obtain polynomial interpolants on arbitrary finite subsets of $\mathbf{R}$, multidimensional polynomial interpolants are much more difficult to obtain and may not exist for arbitrary finite subsets of points in $\mathbf{R^d}$, $d \geq 2$ [4]. In fact, that difficulty has occurred in previous work when using polynomial particular solutions in the DRM [6]. However, if the points are chosen on a rectangular grid in $\mathbf{R^2}$ or a box grid in $\mathbf{R^3}$, then one can find Lagrange interpolants by using the forms of the one-dimensional interpolants discussed previously.

In $\mathbf{R^2}$, we consider interpolating $f(x, y)$ on the set $\{(x_j, y_k)\}$, $0 \leq j \leq m$, $0 \leq k \leq n$. Now let $\{l_j(x)\}_{j=0}^m$ be the Lagrange polynomials for $\{x_j\}_{j=0}^m$ as given by (3.4) and $\{l_k(y)\}_{k=0}^n$ be the corresponding Lagrange polynomials for $\{y_k\}_{k=0}^n$. It is straightforward to verify that

$$q_{m,n}(x, y) = \sum_{k=0}^{n} \sum_{j=0}^{m} f(x_j, y_k) l_j(x) l_k(y) \tag{3.12}$$

interpolates to $f(x, y)$ at $\{(x_j, y_k)\}$, $0 \leq j \leq m$, $0 \leq k \leq n$. That is,

$$q_{m,n}(x_j, y_k) = f(x_j, y_k), \qquad 0 \leq j \leq m, \qquad 0 \leq k \leq n.$$

As in the 1D case, it is convenient to choose $\{x_j\}_{j=0}^m$ and $\{y_k\}_{k=0}^n$ as the images of the pseudo-spectral points in $[a, b] \times [c, d]$. Then using (3.11) twice gives

$$q_{m,n}(x, y) = \sum_{k=0}^{n} \sum_{j=0}^{m} a_{jk} T_j\left(\frac{2x - b - a}{b - a}\right) T_k\left(\frac{2y - d - c}{d - c}\right), \tag{3.13}$$

where

$$a_{jk} = \frac{4}{nm\bar{c}_j\bar{c}_k} \sum_{q=0}^{n} \sum_{p=0}^{m} \frac{f(x_p, y_q)}{\bar{c}_p\bar{c}_q} \cos\left(\frac{\pi pj}{n}\right)\cos\left(\frac{\pi qk}{m}\right). \tag{3.14}$$

The above expansion can be extended to the 3D case in a similar fashion.

IV. PARTICULAR SOLUTIONS FOR POISSON'S EQUATION

In this section, we consider finding a particular solution of Poisson's equation

$$\Delta u_p = f. \tag{4.1}$$

As we discussed in the previous section, f can be approximated by a truncated series of Chebyshev polynomials. We note that the particular solution for monomial right-hand sides for the 2D case are available [6]. However, no such particular solution for monomial right-hand sides are available in the 3D case. A different approach for the 3D case is considered in this section.

A. The 2D Case

A particular solution of

$$\Delta \psi = x^m y^n, \qquad m \geq 0,\, n \geq 0, \tag{4.2}$$

is given by [6]:

$$\psi(x, y) = \begin{cases} \displaystyle\sum_{k=1}^{[(n+2)/2]} (-1)^{k+1} \frac{m!n!x^{m+2k}y^{n-2k+2}}{(m+2k)!(n-2k+2)!}, & \text{for } m \geq n, \\ \displaystyle\sum_{k=1}^{[(m+2)/2]} (-1)^{k+1} \frac{m!n!x^{m-2k+2}y^{n+2k}}{(m-2k+2)!(n+2k)!} & \text{for } m < n, \end{cases} \tag{4.3}$$

where $[s]$ is the largest integer that is less than or equal to s. Chen et al. [16] gave a different approach for deriving particular solutions when the right-hand side of (4.2) is a homogeneous polynomial of degree n.

Theorem 4.1. *A particular solution of*

$$\Delta \psi = \sum_{k=0}^{n} A_k x^{n-k} y^k \tag{4.4}$$

is given by

$$\psi = \sum_{k=0}^{n} P_k x^{n-k+2} y^k \tag{4.5}$$

where

$$P_k = \sum_{m=0}^{[(n-k)/2]} \frac{(-1)^m (k+2m)!(n-k-2m)!}{k!(n-k+2)!} A_{k+2m}, \qquad \textit{for } 1 \leq k \leq n. \tag{4.6}$$

Proof. See [16]. ■

B. The 3D Case

A particular solution of

$$\Delta\psi = x^l y^m z^n \tag{4.7}$$

can be obtained by inspection, similarly to the 2D case:

$$\psi = \sum_{i=1}^{[m/2]+[n/2]+1} x^{l+2i} \sum_{j=\max\{1,i-[n/2]\}}^{\min\{i,[m/2]+1\}} a_{ij} y^{m-2j+2} z^{n-2i+2j}, \tag{4.8}$$

where

$$a_{ij} = \frac{-1}{(l+2i)(l+2i-1)} [(m-2j+4)(m-2j+3)a_{i-1,j-1} + (n-2i+2j+2)(n-2i+2j+1)a_{i-1,j}], \tag{4.9}$$

$$a_{11} = \frac{-1}{(l+2)(l+1)}, \qquad a_{i0} = a_{0j} = 0. \tag{4.10}$$

The particular solution in (4.8) can be verified easily by direct substitution of (4.8) into (4.7).

A more general approach is to find a particular solution when the right-hand side is a homogeneous polynomial of degree n as we have shown in the 2D case.

Let $Q_n(x, y, z)$ be an arbitrary homogeneous polynomial of power $n > 0$; i.e.,

$$Q_n(x, y, z) = \sum_{k=0}^{n} z^k \sum_{m=0}^{n-k} A_{k,m} x^{n-k-m} y^m. \tag{4.11}$$

For the equation

$$\Delta\psi = Q_n(x, y, z), \tag{4.12}$$

we are looking for a particular solution in the form

$$\psi = \sum_{k=0}^{n+2} z^k \sum_{m=0}^{n-k+2} P_{k,m} x^{n-k-m+2} y^m. \tag{4.13}$$

Substituting (4.13) into (4.12) and comparing the coefficients of $\Delta\psi$ with Q_n, we obtain the following system of equations:

$$\begin{aligned} A_{k,m} &= (n-k-m+2)(n-k-m+1)P_{k,m} + (m+2)(m+1)P_{k,m+2} \\ &\quad + (k+2)(k+1)P_{k+2,m}, \qquad k \geq 0, \quad m \geq 0, \quad m+k \leq n. \end{aligned} \tag{4.14}$$

The above system contains $(n+1)(n+2)/2$ equations with $(n+3)(n+4)/2$ unknowns. This means that there are $2n+5$ free parameters, which can be set to zero because of the fact that the particular solution is not unique. The unknowns $P_{k,n}$ in (4.14) can be partitioned into $[(n+4)/2]$ subsets as follows:

$$S_{[(n+2)/2]} = \{P_{k,m} : k+m = n+2 \text{ or } k+m = n+1\} \tag{4.15}$$

$$S_{[n/2]} = \{P_{k,m} : k+m = n \text{ or } k+m = n-1\} \tag{4.16}$$

$$S_{[(n-2)/2]} = \{P_{k,m} : k+m = n-2 \text{ or } k+m = n-3\}. \tag{4.17}$$

$$\vdots \tag{4.18}$$

If n is odd, then $S_0 = \{P_{0,0}, P_{0,1}, P_{1,0}\}$. If n is even, then $S_0 = \{P_{0,0}\}$. Because $S_{[(n+2)/2]}$ contains $2n+5$ elements, all of these elements can be set to zero. In (4.14), we observe that S_j, $0 \leq j \leq [n/2]$, can be expressed in terms of the elements of S_{j+1}. We summarize the above stated procedure into the following algorithm.

Algorithm for finding $P_{k,m}$ $(k \geq 0,\ m \geq 0,\ k+m = l \leq n+2)$:

```
INPUT   $n$ (degree of the homogeneous polynomial)
INPUT   $A_{k,m}$, $k = 0, \ldots, n$; $m = 0, \ldots, n$.
Step 1  For $k = 0, \ldots, n+2$ set $P_{k,n-k+2} = 0$
Step 2  For $k = 0, \ldots, n+1$ set $P_{k,n-k+1} = 0$
Step 3  For $l = n, \ldots, 0$, For $k = 0, \ldots, l$ set $m = l - k$ and
            $P_{k,m} = \frac{A_{k,m} - (m+2)(m+1)P_{k,m+2} - (k+2)(k+1)P_{k+2,m}}{(n-l+2)(n-l+1)}$
OUTPUT  For $k = 0, \ldots, n+2$; $m = 0, \ldots, n-k+2$, Print $P_{k,m}$.
```

We remark that the above algorithm is not valid for a right-hand side polynomial of degree zero. For convenience, in the case of a polynomial of degree zero, $Q_0 = A_{0,0}$, we use as particular solution $A_{0,0}x^2/2$.

Example 1. Let $f(x, y, z) = 3e^{x+y+z}$. f can be approximated by $q_{l,m,n}$ (an extension of the 2D $q_{m,n}$ in (3.11)). Choosing $l = m = n = 4$, we have

$$\begin{aligned} q_{4,4,4}(x, y, z) &= 29.3046 + 29.1781x + 14.6314x^2 + 5.2607x^3 + \cdots + 3.3220x^2y^4z^4 \\ &\quad + 1.1944x^3y^4z^4 + 0.2913x^4y^4z^4. \end{aligned} \tag{4.19}$$

Note that $q_{4,4,4}$ contains 65 terms. Using MATHEMATICA, we can easily extract all the homogeneous terms of same degree from (4.19). To illustrate the algorithm for the particular solutions mentioned above, we choose to extract a polynomial of degree 3; i.e.,

$$Q_3 = 5.2607x^3 - 11.0972xy^2 + 29.0109xyz - 11.0972xz^2.$$

Using the algorithm mentioned above, a particular solution ψ in (4.12) can be obtained as follows:

$$\psi(x, y, z) = 0.6329x^5 - 1.8495x^3y^2 + 4.8351x^3yz - 1.8495x^3z^2.$$

Note that for convenience we only show four decimal digits for the coefficients of $q_{4,4,4}$, Q_3 and ψ.

V. PARTICULAR SOLUTIONS FOR HELMHOLTZ-TYPE EQUATIONS

As we observed in Section B, many important time-dependent problems can be reduced to solving inhomogeneous modified Helmholtz equations of the form

$$\Delta u - \lambda^2 u = f. \tag{5.1}$$

When f is approximated by a polynomial, it is sufficient, as noted in Section IV, to obtain particular solutions for monomial right-hand sides. For completeness, we also consider the case of standard Helmholtz operators, $\Delta + \lambda^2$ as well. In Section A, we consider the 2D case and in Section B, the 3D case.

A. The 2D Case

Theorem 5.1. *Let* $\varepsilon \in \{-1, 1\}$. *A particular solution for*

$$\Delta\psi + \varepsilon\lambda^2\psi = x^m y^n, \qquad m \geq 0,\ n \geq 0, \tag{5.2}$$

is given by

$$\psi(x, y) = \sum_{k=0}^{[m/2]} \sum_{\ell=0}^{[n/2]} \frac{\varepsilon(-\varepsilon)^{k+\ell}(k+\ell)!m!n!x^{m-2k}y^{n-2\ell}}{\lambda^{2k+2\ell+2}k!\ell!(m-2k)!(n-2\ell)!}. \tag{5.3}$$

Proof. See [7]. ■

B. The 3D Case

The following Lemma is necessary for the proof of Theorem 5.3 later.

Lemma 5.2. *A particular solution of the difference equation*

$$B_{j,k,\ell} = B_{j-1,k,\ell} + B_{j,k-1,\ell} + B_{j,k,\ell-1}, \tag{5.4}$$

where $j \geq 0$, $k \geq 0$, $\ell \geq 0$ *and* $(j, k, \ell) \neq (0, 0, 0)$, *is given by*

$$B_{j,k,\ell} = \binom{j+k+\ell}{j+k}\binom{j+k}{k} B_{0,0,0}. \tag{5.5}$$

If any of the indices j, k, ℓ *is negative,* $B_{j,k,\ell} = 0$. *If* $(j, k, \ell) = (0, 0, 0)$, *we have the trivial result* $B_{j,k,\ell} = B_{0,0,0}$.

Proof. Let us assume that the formula is correct when the sum of the indices j, k, and ℓ is n. We will prove the formula for $j + k + \ell = n + 1$, i.e., $(j - 1) + k + \ell = j + (k - 1) + \ell = j + k + (\ell - 1) = n$. Indeed, using Pascal's triangle repeatedly

$$\begin{aligned}
B_{j,k,\ell} &= B_{j-1,k,\ell} + B_{j,k-1,\ell} + B_{j,k,\ell-1} \\
&= \binom{j+k+\ell-1}{j+k-1}\left[\binom{j+k-1}{j-1} + \binom{j+k-1}{j}\right] B_{0,0,0} \\
&\quad + \binom{j+k+\ell-1}{j+k}\binom{j+k}{j} B_{0,0,0} \\
&= \binom{j+k+\ell-1}{j+k-1}\binom{j+k}{j} B_{0,0,0} + \binom{j+k+\ell-1}{j+k}\binom{j+k}{j} B_{0,0,0} \\
&= \binom{j+k}{j}\left[\binom{j+k+\ell-1}{j+k-1} + \binom{j+k+\ell-1}{j+k}\right] B_{0,0,0}.
\end{aligned}$$

Hence,

$$B_{j,k,\ell} = \binom{j+k+\ell}{j+k}\binom{j+k}{j} B_{0,0,0}.$$

■

Theorem 5.3. *A particular solution for*

$$\Delta\psi + \varepsilon\lambda^2\psi = x^p y^q z^r, \qquad p \geq 0,\ q \geq 0,\ r \geq 0, \tag{5.6}$$

is given by

$$\psi(x, y, z) = \sum_{j=0}^{[p/2]} \sum_{k=0}^{[q/2]} \sum_{\ell=0}^{[r/2]} \frac{\varepsilon(-\varepsilon)^{k+\ell}(j+k+\ell)!\,p!\,q!\,r!\,x^{p-2j}y^{q-2k}z^{r-2\ell}}{\lambda^{2j+2k+2\ell+2} j!\,k!\,\ell!\,(p-2j)!\,(q-2k)!\,(r-2\ell)!}, \tag{5.7}$$

where $[s]$ *is the largest integer that is less than or equal to* s.

Proof. Assume ψ to be of the form

$$\psi = \sum_{j=0}^{[p/2]} \sum_{k=0}^{[q/2]} \sum_{\ell=0}^{[r/2]} x^{p-2j}y^{q-2k}z^{r-2\ell} A_{j,k,\ell}, \tag{5.8}$$

where $A_{j,k,\ell}$ are to be determined. We do this by calculating $\Delta\psi + \varepsilon\lambda^2\psi$ setting it equal to $x^p y^q z^r$ and equating the coefficients of corresponding terms. Hence,

$$\psi_{xx} = \sum_{j=0}^{[p/2]} \sum_{k=0}^{[q/2]} \sum_{\ell=0}^{[r/2]} x^{p-2j-2} y^{q-2k} z^{r-2\ell}(p-2j)(p-2j-1)A_{j,k,\ell}$$

$$= \sum_{j=0}^{[p/2]} \sum_{k=1}^{[q/2]} \sum_{\ell=0}^{[r/2]} x^{p-2j} y^{q-2k} z^{r-2\ell}(p-2j+2)(p-2j+1)A_{j-1,k,\ell} \tag{5.9}$$

$$\psi_{yy} = \sum_{j=0}^{[p/2]} \sum_{k=0}^{[q/2]} \sum_{\ell=1}^{[r/2]} x^{p-2j} y^{q-2k} z^{r-2\ell}(q-2k+2)(q-2j+1)A_{j,k-1,\ell} \tag{5.10}$$

$$\psi_{zz} = \sum_{j=1}^{[p/2]} \sum_{k=0}^{[q/2]} \sum_{\ell=0}^{[r/2]} x^{p-2j} y^{q-2k} z^{r-2\ell}(r-2\ell+2)(r-2l+1)A_{j,k,\ell-1}. \tag{5.11}$$

Substituting (5.9)–(5.11) into (5.6) and equating the coefficients of corresponding terms, we obtain

$$\delta_{j,k,\ell} = (p-2j+2)(p-2j+1)A_{j-1,k,\ell} + (q-2k+2)(q-2j+1)A_{j,k-1,\ell} \tag{5.12}$$

$$+ (r-2\ell+2)(r-2\ell+1)A_{j,k,\ell-1} + \varepsilon\lambda^2 A_{j,k,\ell}, \tag{5.13}$$

where for $0 \le j \le [p/2]$, $0 \le k \le [q/2]$, $0 \le \ell \le [r/2]$

$$\delta_{j,k,\ell} = \begin{cases} 1, & j = k = \ell = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{5.14}$$

Now let

$$B_{j,k,\ell} = (p-2j)!(q-2k)!(r-2\ell)!(-\varepsilon\lambda^2)^{j+k+\ell}A_{j,k,\ell}. \tag{5.15}$$

Hence, it follows from (5.14) and (5.15) that $B_{j,k,\ell}$ satisfy the difference equation (5.4):

$$B_{j,k,\ell} = B_{j-1,k,\ell} + B_{j,k-1,\ell} + B_{j,k,\ell-1} + \beta_{j,k,\ell},\ 0 \le j \le \left[\frac{p}{2}\right],\ 0 \le k \le \left[\frac{q}{2}\right],\ 0 \le \ell \le \left[\frac{r}{2}\right], \tag{5.16}$$

where

$$\beta_{j,k,\ell} = \begin{cases} \dfrac{p!q!r!}{\varepsilon\lambda^2}, & j = k = \ell = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\beta = \beta_{0,0,0}$. Also, $B_{-1,k,\ell} = B_{j,-1,\ell} = B_{j,k,-1} = 0$. Hence,

$$\beta = B_{0,0,0} = \beta_{0,0,0} = \frac{p!q!r!}{\varepsilon\lambda^2}. \tag{5.17}$$

From Lemma 5.2,

$$B_{j,k,\ell} = \binom{j+k+\ell}{j+k}\binom{j+k}{j}\beta.$$

From (5.15) and (5.17),

$$(p-2j)!(q-2k)!(r-2\ell)!\lambda^{2j+2k+2\ell}A_{j,k,\ell} = \frac{(j+k+\ell)!}{(j+k)!\ell!}\frac{(j+k)!}{j!k!}\frac{p!q!r!}{\varepsilon\lambda^2}. \tag{5.18}$$

Hence,

$$A_{j,k,\ell} = \frac{p!q!r!(j+k+\ell)!\varepsilon}{j!k!\ell!(p-2j)!(q-2k)!(r-2\ell)!}\frac{(-\varepsilon)^{j+k+\ell}}{\lambda^{2j+2k+2\ell+2}}. \qquad \blacksquare$$

Example 2. The above derived particular solutions can be easily obtained using MATHEMATICA. For the modified Helmholtz operator ($\varepsilon = -1$), and monomial source term x^3y^4 ($m = 3$ and $n = 4$), the particular solution of (5.2) is given by

$$\psi(x, y) = -\frac{432x}{390625} - \frac{24x^3}{15625} - \frac{144xy^2}{15625} - \frac{12x^3y^2}{625} - \frac{6xy^4}{625} - \frac{x^3y^4}{25}. \tag{5.19}$$

Based on the particular solution for a monomial right hand side, we can extend the result to find the particular solution when the right hand side of (5.2) is $T_i(x)T_j(y)$. Let us consider the case $i = 3, j = 4$; then

$$\begin{aligned} T_3(x)T_4(y) &= (-3x + 4x^3)(1 - 8y^2 + 8y^4) \\ &= -3x + 4x^3 + 24xy^2 - 32x^3y^2 - 24xy^4 + 32x^3y^4. \end{aligned} \tag{5.20}$$

The above expansion can be achieved by the MATHEMATICA code $\phi[3, 4]$, where

$$\phi[i, j] := \text{Expand[ChebyshevT}[3, x]\textit{ChebyshevT}[4{,}y]]$$

The monomial terms in (5.20) need to be extracted one at a time so that their particular solutions can be obtained as in (5.19). To extract the coefficients in (5.20), we use the command

$$\text{CoefficientList}[\phi[3, 4], \{x, y\}]$$

to obtain the following matrix

$$\begin{array}{c|ccccc|}
 & 1 & y & y^2 & y^3 & y^4 \\
1 & 0 & 0 & 0 & 0 & 0 \\
x & -3 & 0 & 24 & 0 & -24 \\
x^2 & 0 & 0 & 0 & 0 & 0 \\
x^3 & 4 & 0 & -32 & 0 & 32
\end{array}.$$

For instance, the coefficient of x^3y^2 in (5.20) can be obtained by the command CoefficientList [$\phi[3, 4]$, $\{x, y\}$][[4, 3]]. Consequently, a particular solution of

$$\Delta\Psi - \lambda^2\Psi = T_i(x)T_j(y) \tag{5.21}$$

can be obtained by using the following code:

$$\psi[m\_, n\_, x\_, y\_, \lambda\_] := \sum_{k=0}^{IntegerPart[m/2]} \sum_{l=0}^{IntegerPart[n/2]} \frac{-m!n!(k+\ell)!x^{m-2k}y^{n-2\ell}}{\lambda^{2k+2\ell+2}k!\ell!(m-2k)!(n-2\ell)!}$$

$$\Psi[i\_, j\_, x\_, y\_, \lambda\_] := \sum_{m=1}^{i+1} \sum_{n=1}^{j+1} \text{CoefficientList}[\phi[i, j], \{x, y\}][[m, n]]\psi[m-1, n-1, x, y, \lambda]$$

For $i = 3, j = 4, \lambda = 5$, the particular solution Ψ in (5.21) can be obtained using $\Psi[3, 4, x, y, 5]$ that is equal to

$$\frac{1}{390625}x(21651 - 190200y^2 + 255000y^4 - 100x^2(417 - 2600y^2 + 5000y^4)).$$

VI. THE METHOD OF FUNDAMENTAL SOLUTIONS

For completeness, we briefly introduce the method of fundamental solutions (MFS), a boundary meshless approach for solving the homogeneous equation solution (2.5)–(2.6). In the MFS, we embed the boundary of the domain into an auxiliary boundary $\partial\Omega_A$, that is $\partial\Omega_A \supset \partial\Omega$. In general, we choose $\partial\Omega_A$ as a circle in 2D and a sphere in 3D [9, 17]. We then place the source points on $\partial\Omega_A$. In general the source points are evenly distributed on a sphere containing the domain Ω. The purpose of moving the source points outside of the domain Ω is to avoid the singularities of the fundamental solutions of the operator.

Let $\{P_j\}_{j=1}^m$ be a set of source points lying on the auxiliary boundary $\partial\Omega_A$. We approximate the solution $v(P)$ of (2.5)–(2.6) by a function of the form [9, 18]

$$v_m(P) = \sum_{j=1}^{m} c_jG(P, P_j), \qquad P_j \in \partial\Omega_A, \tag{6.1}$$

where $G(P, P_j)$ is a fundamental solution given by

$$G(P, P_j) = \begin{cases} \dfrac{1}{2\pi} \log\|P - P_j\|, \; P, P_j \in \mathbf{R}^2, \\ & \text{for } L = \Delta, \\ \dfrac{-1}{4\pi\|P - P_j\|}, \; P, P_j \in \mathbf{R}^3, \end{cases}$$

$$G(P, P_j) = \begin{cases} \dfrac{1}{2\pi} K_0(\lambda\|P - P_j\|), & P, P_j \in \mathbf{R}^2, \\ & \qquad \text{for } L = \Delta - \lambda^2, \\ \dfrac{-1}{4\pi\|P - P_j\|} \exp(-\lambda\|P - P_j\|), & P, P_j \in \mathbf{R}^3, \end{cases}$$

and $\|\cdot\|$ is the Euclidean norm. For collocation, we need to choose a set of points $\{Q_k\}_{k=1}^m$ on $\partial\Omega$. Applying the boundary conditions of (2.6) to (6.1), we obtain the following system of equations

$$\sum_{j=1}^{m} c_j G(Q_k, P_j) + c = g(Q_k) - Bu_p(Q_k), \qquad \text{for } k = 1, 2, \ldots, m. \tag{6.2}$$

The above linear system of equations can be solved for $\{c_j\}_{j=1}^m \cup \{c\}$ by a linear solver. Bogomolny [18] showed that the auxiliary boundary $\partial\Omega_A$ can be taken as a circle in 2D (sphere in 3D) and $\{P_j\}_{j=1}^m$ equally distributed. As indicated by Bogomolny [18], the larger the radius of the source circle (sphere), the better the approximation to be expected. In this case, the resulting matrix in (6.2) becomes extremely ill-conditioned. We refer the readers to recent work where this ill-conditioning is studied [19, 20].

Approximations v_m to v and $\partial v_m/\partial n$ to $\partial v/\partial n$ are then given by

$$v_m(P) = \sum_{j=1}^{m} c_j G(P, P_j) + c + u_p, \qquad P \in \bar{\Omega}, \tag{6.3}$$

$$\frac{\partial}{\partial n} v_m(P) = \sum_{j=1}^{m} c_j \frac{\partial}{\partial n} G(P, P_j) + \frac{\partial}{\partial n} u_p, \qquad P \in \bar{\Omega}, \tag{6.4}$$

VII. NUMERICAL RESULTS

To demonstrate the effectiveness of the proposed method, we give three examples using the symbolic computational software package MATHEMATICA.

Example 3. Consider the 2D Poisson problem

$$\Delta u(x, y) = 2e^{x-y}, \qquad (x, y) \in \Omega, \tag{7.1}$$

$$u(x, y) = e^{x-y} + e^x \cos y, \qquad (x, y) \in \partial\Omega, \tag{7.2}$$

where $\Omega = [-1, 1]^2$. The exact solution is

TABLE I. Errors $u_p(x, y) - u_{p,n}(x, y)$.

x	y	$u_p(x, y)$	$u_p(x, y) - u_{p,n}(x, y)$ $n = 4$	$n = 6$	$n = 8$
0.0	0.0	0.0	0.0	0.0	0.0
0.6262	0.0	0.09643	2.567×10^{-4}	-1.506×10^{-6}	4.106×10^{-9}
0.1740	0.1740	0.00183	7.484×10^{-6}	-6.407×10^{-8}	-2.882×10^{-10}
0.3728	0.3728	0.13579	-4.504×10^{-4}	3.552×10^{-7}	-1.359×10^{-9}
0.0	0.2071	0.04007	-1.209×10^{-4}	1.030×10^{-7}	-4.480×10^{-10}
−0.3728	0.3728	0.11284	-8.363×10^{-4}	6.432×10^{-7}	-2.414×10^{-9}

$$u(x, y) = e^{x-y} + e^x \cos y.$$

To approximate the forcing term using Chebyshev interpolation, we choose $m = n$ in (3.12), since the solution domain is a square. The particular solution can be obtained as discussed in Section A. We note that the particular solution is not unique. In this example, the exact particular solution $u_p(x, y)$ was obtained by taking $n = 12$. The results in Table I show the error $u_p(x, y) - u_{p,n}(x, y)$ at six selected points in the domain. This indicates that the computational cost of evaluating approximate particular solutions using the current approach is not necessarily high.

The MFS [9, 18] was applied to evaluate the homogeneous solution. In the MFS, we choose 32 uniformly distributed collocation points on the boundary. The same number of source points on the fictitious boundary, a circle with radius 6 and center (0, 0), were also chosen. The approximate solution $\hat{u}$ and its derivatives $\hat{u}_x$ and $\hat{u}_y$ were evaluated at 100 evenly distributed grid points. Moreover, the second derivative can also be approximated accurately. The overall absolute maximum errors for various values of n are given in Table II. The numerical results are in excellent agreement with the exact solution.

In order to demonstrate that the solution process can be applied to an irregular domain, we changed the physical domain $[-1, 1] \times [-1, 1]$ of the above problem to the shape of peanut. Define

$$R(\theta) = \sqrt{\cos(2\theta) + \sqrt{1.1 - \sin^2(2\theta)}}$$

$$\partial\Omega = \{\mathrm{R}(\theta)(\cos\theta, \sin\theta) : 0 \leq \theta \leq 2\pi\}. \tag{7.3}$$

The picture of (7.3), often called the Oval of Cassini in the mathematical literature, is shown in Fig. 1.

TABLE II. The absolute maximum errors of u, u_x, u_y and Δu.

n	$\|u - \hat{u}\|_\infty$	$\|u_x - \hat{u}_x\|_\infty$	$\|u_y - \hat{u}_y\|_\infty$	$\|\Delta\hat{u} - f\|_\infty$
4	1.678×10^{-4}	1.299×10^{-3}	8.767×10^{-4}	5.126×10^{-3}
5	1.861×10^{-5}	7.039×10^{-5}	6.667×10^{-5}	4.162×10^{-4}
6	1.028×10^{-6}	4.300×10^{-6}	4.047×10^{-6}	3.117×10^{-5}
7	3.418×10^{-8}	2.079×10^{-7}	1.907×10^{-7}	1.577×10^{-6}
8	1.577×10^{-9}	1.057×10^{-8}	2.511×10^{-8}	1.351×10^{-7}
9	3.882×10^{-11}	2.606×10^{-9}	7.076×10^{-10}	3.752×10^{-9}

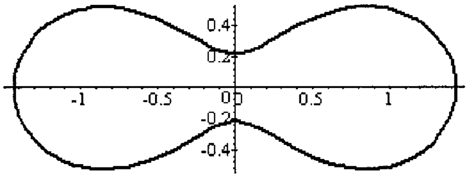

FIG. 1. Oval of Cassini.

To evaluate the particular solution, we imbedded the peanut shaped domain into a rectangular box $[-1.5, 1.5] \times [-0.6, 0.6]$. The transformation of the domain can be handled by using (3.13)–(3.14).

To evaluate the homogeneous solution, we use the MFS. Thirty-five points were evenly distributed (in terms of the angle) on the boundary and the same number of points were chosen on the fictitious boundary, a circle with center (0, 0) and radius 8. In order to compare the results to those of [5] and [21], we examined the solution at the eight selected points. The numerical results are shown in Table III. Comparing with the approach in [5], our approach using Chebyshev interpolation is more general than the Taylor series expansion in [5] even though the numerical accuracy is comparable. The accuracy of our results is certainly superior to the results in [21], using multiquadric radial basis functions.

Example 4. In this example, we consider the following 3D Poisson problem

$$\Delta u(x, y, z) = 3e^{x+y+z}, \qquad (x, y, z) \in \Omega, \tag{7.4}$$

$$u(x, y, z) = e^{x+y+z}, \qquad (x, y, z) \in \partial\Omega, \tag{7.5}$$

where $\Omega = [-1, 1]^3$. The exact solution is given by $u(x, y, z) = e^{x+y+z}$. Because the solution domain is a cube, we choose the same Chebyshev nodes in each axis direction as in the 2D case in Example 3. In the 3D case, the number of function evaluations of the particular solution is significantly higher than in the 2D case. Hence, we only perform numerical evaluation of particular solutions for $n \leq 6$. For the evaluation of the homogeneous solution, we choose 200 uniformly distributed collocation points on the surface of the cube and the same number of source points on the surface of a sphere with radius 6 and center at the origin. We choose 100 random testing points in the solution domain. The absolute maximum errors for various values of n are shown in Table IV. The numerical results are also very accurate.

Example 5. Consider the following modified Helmholtz equation

TABLE III. Absolute maximum errors in approximation to $u(x, y)$.

x	y	$n = 5$	$n = 7$	$n = 9$	$n = 11$
0	0	4.959E-5	0	1.629E-9	1.818E-12
0.6262	0	3.230E-5	1.718E-7	7.408E-11	7.695E-12
1.3419	0	7.207E-5	7.455E-5	4.857E-7	2.565E-9
0.174	0.174	2.285E-6	3.432E-7	1.488E-9	1.227E-11
0.3728	0.3728	6.615E-5	9.329E-7	5.671E-9	2.659E-11
0	0.2071	6.659E-6	1.331E-7	8.679E-11	1.066E-11
−0.3728	0.3728	3.965E-5	4.071E-7	1.786E-9	5.041E-11
−1.3419	0	7.076E-2	7.909E-4	5.136E-6	2.214E-8

TABLE IV. The absolute maximum errors of u.

n	$\|u - \hat{u}\|_\infty$
4	3.153×10^{-4}
5	4.110×10^{-5}
6	5.761×10^{-6}

$$(\Delta - \lambda^2)u(x, y) = (1 - \lambda^2)(e^x + e^y), \qquad (x, y) \in \Omega, \tag{7.6}$$

$$u(x, y) = e^x + e^y, \qquad (x, y) \in \partial\Omega, \tag{7.7}$$

where $\Omega = [-1, 1]^2$. The exact solution is given by $u(x, y) = e^x + e^y$. The procedure for evaluating particular solutions and the homogeneous solution are the same as in Example 3. The maximum absolute error using various n and wave number λ are given in Table V. In general, the solution for (7.6)–(7.7) is difficult to approximate for high wave numbers. We denote $\hat{u}$ the approximate solution of u. As shown in Table V, high wave numbers can be handled by using a large number of Chebyshev nodes. In Table VI, we observe that the approximate derivative $\hat{u}_y$ deteriorates for high wave numbers. However, when n is chosen high enough, we can still obtain excellent approximation of the derivative. The numerical results for $\hat{u}_x$ are similar to $\hat{u}_y$ and will not be shown here.

VIII. CONCLUSIONS

Chebyshev interpolation has been investigated to approximate the right-hand side of Poisson and Helmholtz-type equations in 2D and 3D. Particular solutions of these types of differential operators have been obtained in previous and current articles. In contrast to the implementation of the particular solutions using radial basis functions, no matrix inversion is required and the difficulty of ill-conditioning for a large amount of interpolation points is alleviated. With the powerful features of symbolic software packages such as MAPLE and MATHEMATICA, the implementation of the proposed algorithm becomes feasible. However, it is not clear how the numerical implementation can be carried out on other platforms such as Fortran and C++. The evaluation of the particular solutions for the 3D case is still computationally intensive. The improvement of the efficiency in the 3D case is a subject of future research. Furthermore, our proposed numerical scheme can be easily extended to time-dependent problems and is currently under investigation.

TABLE V. The absolute maximum errors $\|u - \hat{u}\|_\infty$.

n	$\lambda = 10$	$\lambda = 30$	$\lambda = 50$	$\lambda = 100$
4	1.747×10^{-3}	6.399×10^{-3}	0.236	553.068
5	1.133×10^{-4}	1.516×10^{-4}	2.692×10^{-4}	2.692×10^{-3}
6	7.401×10^{-6}	7.139×10^{-5}	2.070×10^{-3}	4.266
7	3.745×10^{-7}	2.710×10^{-6}	8.908×10^{-5}	0.180
8	2.075×10^{-8}	2.613×10^{-7}	6.365×10^{-6}	2.692×10^{-2}
9	2.109×10^{-9}	2.770×10^{-9}	1.431×10^{-8}	2.692×10^{-7}

TABLE VI. The absolute maximum errors $\|u_y - \hat{u}_y\|_\infty$.

n	$\lambda = 10$	$\lambda = 30$	$\lambda = 50$
4	2.035×10^{-2}	13.75	5344.27
5	8.203×10^{-4}	0.118	68.149
6	1.549×10^{-4}	7.139×10^{-3}	0.119
7	2.316×10^{-6}	2.710×10^{-4}	0.313
8	3.180×10^{-6}	2.613×10^{-4}	0.1383
9	2.440×10^{-8}	2.309×10^{-5}	1.87×10^{-2}
10	2.294×10^{-9}	2.647×10^{-8}	7.260×10^{-7}

The authors thank the referees for their helpful comments.

References

1. D. Nardini and C. A. Brebbia, A new approach to free vibration analysis using boundary elements, C. A. Brebbia, editor, Boundary element methods in engineering, Proc. 4th Int. Sem., Springer-Verlag, New York, 1982, pp 312–326.
2. P. W. Partridge, C. A. Brebbia, and L. C. Wrobel, The dual reciprocity boundary element method, Computational Mechanics Publications, Southampton, Boston, 1992.
3. A. S. Muleshkov, M. A. Golberg, and C. S. Chen, Particular solutions of Helmholtz-type operators using higher order polyharmonic splines, Comp Mech 23 (1999), 411–419.
4. M. A. Golberg and C. S. Chen, Discrete projection methods for integral equations, Computational Mechanics Publications, Southampton, 1997.
5. K. E. Atkinson, The numerical evaluation of particular solutions for Poisson's equation, IMA J Numer Anal 5 (1985), 319–338.
6. A. H.-D. Cheng, O. Lafe, and S. Grilli, Dual reciprocity BEM based on global interpolation functions, Eng Anal Boundary Elements 13 (1994), 303–311.
7. A. S. Muleshkov, C. S. Chen, M. A. Golberg, and A. H.-D. Cheng, Analytic particular solutions for inhomogeneous Helmholtz-type equations, S. N. Atluri, F. W. Brust, editors, Advances in computational engineering and sciences, Tech Science Press, 2000, pp 27–32.
8. A. Poullikkas, A. Karageorghis, and G. Georgiou, The method of fundamental solutions for inhomogeneous elliptic problems, Comp Mech 22 (1998), 100–107.
9. M. A. Golberg and C. S. Chen, The method of fundamental solutions for potential, Helmholtz and diffusion problems, M. A. Golberg, editor, Boundary integral methods: numerical and mathematical aspects, WIT Press and Computational Mechanics Publications, Boston, Southampton, 1999, pp 103–176.
10. R. Chapko and R. Kress, Rothe's method for the heat equation and boundary integral equations, J Integ Eq Appl 9 (1997), 47–68.
11. C. S. Chen, M. A. Golberg, and A. S. Muleshkov, The method of fundamental solutions for time-dependent equations, C. S. Chen, C. A. Brebbia, and D. W. Pepper, editors, Boundary element technology XIII, WIT Press, Boston, Southampton, 1999, pp 377–386.
12. C. Canuto, M. Y. Hussai, A. Quarteroni, and T. A. Zang, Spectral methods in fluid dynamics, Springer-Verlag, New York, 1988.
13. J. P. Boyd, Spectral methods in fluid dynamics, Second Edition, Dover Publications, New York, 2001.
14. C. Bernardi and Y. Maday, Spectral methods, P. G. Ciarlet and J.-L. Lions, editors, Handbook of numerical analysis, Vol. V, 1997, pp 209–485.
15. C. Canuto and A. Quarteroni, Approximation results for orthogonal polynomials in Sobolev spaces, Math Comp 38 (1982), 67–86.

16. C. S. Chen, M. A. Golberg, and A. S. Muleshkov, The numerical evaluation of particular solutions for Poisson's equation—a revisit, C. A. Brebbia, H. Power, editors, Boundary element methods XXI, WIT Press, Southampton, 1999, pp 313–322.

17. G. Fairweather and A. Karageorghis, The method of fundamental solutions for elliptic boundary value problems, Adv Comp Math 9 (1998), 69–95.

18. A. Bogomolny, Fundamental solutions method for elliptic boundary value problems, SIAM J Numer Anal 22 (1998), 644–669.

19. K. Balakrishnan and P. A. Ramachandran, The method of fundamental solutions for linear diffusion-reaction equations, Math Comput Modelling 31 (2000), 221–237.

20. Y. S. Smyrlis and A. Karageorghis, Some aspects of the method of fundamental solutions for certain harmonic problems, J Sci Comput 16 (2001), 341–371.

21. M. A. Golberg, C. S. Chen, and S. R. Karur, Improved multiqadric approximation for partial differential equations, Eng Anal Boundary Elem 18 (1996), 9–17.